Velocity and Pressure Fields Induced by Spheres in an Unbounded Fluid

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We propose a procedure for the determination of the time-dependent velocity and pressure fields of an unbounded incompressible viscous fluid in an external force field induced by an arbitrary number of spheres moving and rotating in it as well as the forces and torques exerted by the fluid on the particles. Within the completely linearized scheme, we express the velocity and pressure fields of the fluid in terms of induced surface force densities and derive the explicit form for all quantities contained in these relations not imposing any additional restrictions on the size of particles, distances between them, and the frequency range. We show the incorrectness of similar results obtained earlier by several authors because these results are expressed in terms of nonexistent inverse tensors. We explain the reasons leading to this and propose a procedure for the elimination of divergent quantities. In the stationary case, using the proposed procedure, we obtained the translational, rotational, and coupled friction and mobility tensors for a system containing an arbitrary number of spheres up to, respectively, the second, forth, and third orders in the dimensionless parameter equal to the ratio of a typical radius of a sphere to a typical distance between two spheres. In various particular cases, the results obtained in the present paper agree with the well-known results derived by other methods.

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I. INTRODUCTION

Investigation of hydrodynamic interactions between spheres immersed in an incompressible viscous fluid is of considerable interest for a wide class of problems of physics of suspensions and colloidal crystals. Due to the long-range nature of the hydrodynamic interactions, their account is essentially important for the study of these systems, for which many-particle interactions should be taken into account. As a rule, for the study of these interactions, the linearized Navier–Stokes equation for the fluid is used [1]. Usually, the problem is reduced to the determination of the forces and torques exerted by the fluid on particles moving and rotating in it with given translational and rotational velocities. According to the classical approach used for one particle [2–6], to find these forces and torques, first, it is necessary to determine the velocity and pressure fields of the fluid induced by particles moving and rotating in it. However, in the case of several particles, this problem is extremely complicated. The corresponding exact relations for the forces and torques exerted on particles due to hydrodynamic interactions between them were derived only for the particular stationary case of two spheres moving along the line passing through their centers with equal [7] or different velocities [8] and perpendicular to this line and rotating along the axis perpendicular to the direction of motion of the spheres and the line connecting their centers [9,10].

For this reason, several methods aimed at the determination of approximate solutions have been developed. Among these methods, there is the well-known classical method of reflections first proposed by Smoluchowski for analysis of the forces exerted by the fluid on n spheres moving in it with constant velocities. Later, this method was used for the solution of many stationary problems of hydrodynamic interactions of particles (mainly, two particles) in an incompressible viscous fluid (the detailed review of results is given in [1]) including also the cases of permeable spheres [11–13] and mixed slip-stick boundary conditions at the surfaces of the spheres [14–18]. The corresponding results are presented in the form of power series in the dimensionless parameter σ equal to the ratio of a typical radius of a sphere to a typical distance between two spheres calculated to a certain order of this parameter.

Since the method of reflections and its further modifications are based on the solution of the corresponding boundary-value problems, their direct application to the solution of nonstationary many-particle problems seems to be rather problematical. Moreover, even in the stationary case, the procedure of determination of the velocity field of the fluid induced by particles becomes essentially more complicated with increase in the number of particles. For this reason, the methods of reflections are usually used for two spheres in the stationary case. In [19], Mazur and Bedeaux proposed a new method (called the method of induced forces) for the determination of the force exerted by the fluid on a single sphere moving with time-dependent velocity in the case where the fluid moves with nonstationary and nonhomogeneous velocity. Later, this method was developed for the determination of the friction [20] and mobility tensors [21] in the case of the stationary fluid containing an arbitrary number of particles, furthermore, the results for the mobility tensors were generalized to the nonstationary case [22]. At finite frequencies, the expressions for the mobility tensors are obtained up to the third order in two dimensionless parameters, namely, the parameter σ, which is typical of the stationary case, and the parameter κ, which is proportional to the ratio of a typical radius of a sphere to the penetration depth of transverse waves. The method of induced forces essentially differs from the method of reflections because it is not based on the necessity of the knowledge of the explicit form for the fluid velocity induced by particles.

Other methods for the solution of problems of hydrodynamic interactions between particles in the fluid were proposed in [23–29]. Despite the fact that the solution of nonstationary many-particle problems in [25,26] is also based on the introduction of unknown induced forces, this approach essentially differs from the method of induced forces used in [20–22]. Indeed, within the framework of this approach, the forces and torques exerted by the fluid on particles as well as the velocity field of the fluid induced by these particles are expressed in terms of the introduced induced surface forces, the explicit form for which was determined up to the third order of two dimensionless parameters mentioned above, while the method of induced forces in [20–22] is developed in such a way that the mobility and friction tensors (not the velocity field of the fluid that cannot be found with the use of this method) are found without determination of the explicit form of the introduced induced surface forces. However, the final results presented in [25,26] are expressed in terms of multiindex tensors, the explicit expressions for which are not given and that are only defined as tensors inverse to certain rather complicated multiindex tensors. For this reason, the question of the agreement between the results obtained in [25,26] and [22] is open. Furthermore, this problem remains unsolved even for the stationary case. Even in the simplest case of a single sphere rotating with constant angular velocity in a fluid, the fluid velocity followed from the general relations given in [25,26] is also expressed in terms of the inverse tensor and, hence, cannot be reduced to the classical result for the fluid velocity induced by a rotating sphere [1,5] without the knowledge of the explicit form for the inverse tensor.

The aim of the present paper is to develop a method for the solution of various problems (both stationary and nonstationary) of hydrodynamic interactions between any number of particles immersed in an unbounded incompressible viscous fluid in an arbitrary force field such that, as opposed to [25,26], the results obtained by this method (the velocity and pressure fields of the fluid induced by these particles, the forces and torques exerted by the fluid on the particles, various mobility tensors, etc.) coincide with the corresponding results obtained in particular cases by other methods as well as to explain the reasons that do not enable one to represent the results given in [25,26,30] in the conventional form.

In Sec. 2, we reduce the problem of n spheres in an unbounded incompressible viscous fluid in an arbitrary force field to an equivalent problem for this fluid without particles in an efficient force field. Unlike the generally excepted procedure of the zeroth extension of the external force [19,20,23–26] and the fluid pressure [19–22] to the domains occupied by the particles, we use another extensions for these quantities and discuss possible problems connected with different methods of extensions of the required quantities. We give relations for the total forces and torques exerted by the fluid and force fields on particles.

In Sec. 3, within the framework of the completely linearized scheme (both with respect to the fluid velocity and the velocities of the particles), we express the required distributions of the fluid velocity and pressure in terms of the unknown induced surface forces distributed over the surfaces of the particles. The results are obtained without imposing any additional restrictions on the size of particles, distances between them, and frequency range. All required quantities contained in the relations for the velocity and pressure fields of the fluid are given in the explicit form in terms of special functions of dimensionless parameters. We derive the system of algebraic linear equations in the unknown harmonics of the induced surface force densities in Sec. 4.

In Sec. 5, we consider the stationary case. Taking into account the obtained explicit form of the quantities in the system of algebraic equations, which is similar to the system given in [30], we solve this system by the method of successive approximations using the system of noninteracting particles as a zero approximation. In this approximation, we show that the determinant of the system of equations corresponding to the harmonics with $l = 1$ is equal to zero. Therefore, the inverse tensor, in terms of which the relations for the forces and torques exerted by the fluid on the particles given in [30] for the stationary case as well as analogous results and the velocity field of the fluid induced by the particles given in [25,26] for the nonstationary case, does not exist. We analyze the reasons for this fact and show that this means that the required induced surface forces can be determined only up to arbitrary potential components that have no influence on the fluid velocity. We formulate the additional conditions that enable one to uniquely determine the induced surface force densities. Using the proposed procedure, we obtained the translational, rotational, and coupled friction and mobility tensors for a system containing an arbitrary number of spheres up to the second, forth, and third orders in the dimensionless parameter σ, respectively, which in various particular cases, agree with the well-known results obtained by other methods. Within the framework of the considered approach, we formulate an algorithm for the determination of the velocity and pressure fields of the fluid as well as the forces and torques exerted by the fluid on particles corresponding to a given power of the parameter σ. The details of calculation of several integrals of the products of three and two Bessel functions necessary for determination of the hydrodynamic interaction tensors are given in Appendix.

II. GENERAL RELATIONS

We consider an unbounded incompressible viscous fluid in a certain external force field. In the linear approximation, the fluid is described by the linearized Navier–Stokes equation

$$\rho \frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t} + \operatorname{div} \boldsymbol{P}(\boldsymbol{r},t) = \boldsymbol{F}^{ext}(\boldsymbol{r},t) \tag{2.1}$$

and the continuity equation

$$\operatorname{div} \boldsymbol{v}(\boldsymbol{r},t) = 0. \tag{2.2}$$

Here, $\boldsymbol{P}(\boldsymbol{r},t)$ is the stress tensor of the fluid with the components

$$P_{ij}(\boldsymbol{r},t) = \delta_{ij}\, p(\boldsymbol{r},t) - \eta \left(\frac{\partial \boldsymbol{v}_i(\boldsymbol{r},t)}{\partial r_j} + \frac{\partial \boldsymbol{v}_j(\boldsymbol{r},t)}{\partial r_i} \right), \qquad i,j = x,y,z, \tag{2.3}$$

$p(\boldsymbol{r},t)$ and $\boldsymbol{v}(\boldsymbol{r},t)$ are the hydrostatic pressure and velocity fields of the fluid, ρ and η are its density and viscosity, respectively, δ_{ij} is the Kronecker symbol, and $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ is the external force acting on a unit volume of the fluid.

We represent the external force $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ as a superposition of the potential $\boldsymbol{F}^{(p)ext}(\boldsymbol{r},t)$ and solenoidal $\boldsymbol{F}^{(sol)ext}(\boldsymbol{r},t)$ components

$$\boldsymbol{F}^{ext}(\boldsymbol{r},t) = \boldsymbol{F}^{(p)ext}(\boldsymbol{r},t) + \boldsymbol{F}^{(sol)ext}(\boldsymbol{r},t), \tag{2.4}$$

$$\boldsymbol{F}^{(p)ext}(\boldsymbol{r},t) = -\rho \boldsymbol{\nabla} \varphi(\boldsymbol{r},t), \tag{2.5}$$

where $\varphi(\boldsymbol{r},t)$ is the potential of the external conservative force.

Assuming that the quantities in Eqs. (2.1)–(2.3) can be expanded in the Fourier integral of the form

$$A(\boldsymbol{r},t) = \frac{1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} d\omega \int\limits_{-\infty}^{\infty} d\boldsymbol{k}\, A(\boldsymbol{k},\omega) \exp(i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)), \tag{2.6}$$

we represent the frequency Fourier transform of the solution of Eqs. (2.1)–(2.3) as follows:

$$\boldsymbol{v}(\boldsymbol{r},\omega) = \frac{1}{(2\pi)^3 \eta} \int\limits_{-\infty}^{\infty} d\boldsymbol{k}\, \frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{r})}{k^2 + \kappa^2}\, \boldsymbol{F}^{(sol)ext}(\boldsymbol{k},\omega) + \boldsymbol{v}^{inf}(\omega), \tag{2.7}$$

$$\begin{aligned} \rho(\boldsymbol{r},\omega) &= -\frac{i}{(2\pi)^3} \int\limits_{-\infty}^{\infty} d\boldsymbol{k}\, \frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{r})}{k^2}\, \boldsymbol{k}\cdot\boldsymbol{F}^{ext}(\boldsymbol{k},\omega) + \rho^{inf}(\omega) \\ &= -\rho\varphi(\boldsymbol{r},\omega) + \rho^{inf}(\omega), \end{aligned} \tag{2.8}$$

where

$$\boldsymbol{F}^{(sol)ext}(\boldsymbol{k},\omega) = (\boldsymbol{I} - \boldsymbol{n}_k\boldsymbol{n}_k)\cdot\boldsymbol{F}^{ext}(\boldsymbol{k},\omega) \tag{2.9}$$

$\boldsymbol{I}$ is the unit tensor, $\boldsymbol{n}_k = \boldsymbol{k}/k$ is the unit vector directed along the vector $\boldsymbol{k}$, $\kappa = \sqrt{\omega/(i\nu)} = (1 - i\,\mathrm{sign}\,\omega)/\delta$, $\delta = \sqrt{2\nu/|\omega|}$ is the depth of penetration of a plane transverse wave of frequency ω created by an oscillating solid surface into the fluid [5], $\nu = \eta/\rho$ is the kinematic viscosity of the fluid, $\boldsymbol{v}^{inf}(\omega) = 2\pi\delta(\omega)\boldsymbol{v}^{inf}$ and $\rho^{inf}(\omega) = 2\pi\delta(\omega)\rho^{inf}$, where $\delta(\omega)$ is the Dirac delta-function, $\boldsymbol{v}^{inf}$ and ρ^{inf} are certain constant velocity and pressure at infinity defined by the condition for the fluid at infinity where the external force field is absent. According to relation (2.7), the fluid velocity is determined only by the solenoidal component of the external force, which is the natural consequence of the continuity equation (2.2). The fluid pressure is determined by the potential component of the external force.

Let, in the considered fluid, N homogeneous macroscopic spheres of radii a_α and masses m_α, where $\alpha = 1, 2, \ldots, N$, be present. At the time t, the position of the center of sphere α is defined by the radius vector $\boldsymbol{R}_\alpha(t)$ relative to the fixed Cartesian coordinate system with origin at the point O (in what follows, the system O). In parallel with this fixed system, we also introduce N local moving Cartesian coordinate systems with origins O_α at the centers of spheres coinciding with their centers of mass (in what follows, systems O_α) so that sphere α does not move relative to system O_α. Therefore, the radius vector $\boldsymbol{r}$ of any point of the space can be represented in the form $\boldsymbol{r} = \boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha$, where $\boldsymbol{r}_\alpha$ is the radius vector of this point relative to the local system O_α. The motion of the spheres is described by the equations

$$m_\alpha \frac{d\boldsymbol{U}_\alpha(t)}{dt} = \boldsymbol{F}_\alpha^{tot}(t), \tag{2.10}$$

$$I_\alpha \frac{d\boldsymbol{\Omega}_\alpha(t)}{dt} = \boldsymbol{T}_\alpha^{tot}(t), \tag{2.11}$$

where

$$\boldsymbol{U}_\alpha(t) = \frac{d\boldsymbol{R}_\alpha(t)}{dt}$$

and $\boldsymbol{\Omega}_\alpha(t)$ are, respectively, the translational and angular velocities of sphere α, $I_\alpha = (2/5)m_\alpha a_\alpha$ is its moment of inertia, $\boldsymbol{F}_\alpha^{tot}(t)$ is the total force and $\boldsymbol{T}_\alpha^{tot}(t)$ is the total torque (here and in what follows, all torques corresponding to particle α are considered relative to its center) acting on sphere α

$$\boldsymbol{F}_{\alpha}^{tot}(t) = \boldsymbol{F}_{\alpha}^{ext}(t) + \boldsymbol{F}_{\alpha}^{f}(t), \tag{2.12}$$

$$\boldsymbol{T}_{\alpha}^{tot}(t) = \boldsymbol{T}_{\alpha}^{ext}(t) + \boldsymbol{T}_{\alpha}^{f}(t), \tag{2.13}$$

where $\boldsymbol{F}_{\alpha}^{ext}(t)$ and $\boldsymbol{T}_{\alpha}^{ext}(t)$ are the force and torque acting on sphere α due to the external force field and $\boldsymbol{F}_{\alpha}^{f}(t)$ and $\boldsymbol{T}_{\alpha}^{f}(t)$ are the force and torque exerted by the fluid on sphere α

$$\boldsymbol{F}_{\alpha}^{f}(t) = -\int\limits_{S_{\alpha}(t)} \boldsymbol{P}(\boldsymbol{r},t)\cdot\boldsymbol{n}_{\alpha}\, dS_{\alpha}, \tag{2.14}$$

$$\boldsymbol{T}_{\alpha}^{f}(t) = -\int\limits_{S_{\alpha}(t)} \Big((\boldsymbol{r}-\boldsymbol{R}_{\alpha}(t))\times\boldsymbol{P}(\boldsymbol{r},t)\Big)\cdot\boldsymbol{n}_{\alpha}\, dS_{\alpha}, \tag{2.15}$$

$S_{\alpha}(t)$ is the surface of sphere α at the time t and $\boldsymbol{n}_{\alpha}$ is the outward unit vector to this surface.

The fluid is described by the same equations (2.1)–(2.3) as in the case of the absence of particles but defined only in the domain outside the spheres $r_{\alpha} \geq a_{\alpha}, \quad \alpha = 1, 2\ldots, N$, where $r_{\alpha} \equiv |\boldsymbol{r}_{\alpha}| = |\boldsymbol{r} - \boldsymbol{R}_{\alpha}(t)|$.

The problem is to determine the velocity and pressure fields of the fluid induced by moving particles in it as well as the forces and torques exerted by the fluid on the particles. For the solution of this problem, we use the method of induced forces proposed in [19] and developed in [20–22]. We extend the quantities $p(\boldsymbol{r},t)$, $\boldsymbol{v}(\boldsymbol{r},t)$, and $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ defined in Eqs. (2.1) and (2.2) for $r_{\alpha} \geq a_{\alpha}$ to the domains $r_{\alpha} < a_{\alpha}$ occupied by the spheres as follows:

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{U}_{\alpha}(\boldsymbol{r},t) = \boldsymbol{U}_{\alpha}(t) + (\boldsymbol{\Omega}_{\alpha}(t)\times\boldsymbol{r}_{\alpha}), \quad r_{\alpha} < a_{\alpha}, \tag{2.16}$$

$$p(\boldsymbol{r},t) = -\rho\varphi(\boldsymbol{r},t) + p^{inf} - \rho\left(\boldsymbol{r}\cdot\frac{\partial\boldsymbol{U}_{\alpha}(t)}{\partial t}\right), \quad r_{\alpha} < a_{\alpha}. \tag{2.17}$$

We extend the external force field $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ given for $r_{\alpha} \geq a_{\alpha}$ to the domains $r_{\alpha} < a_{\alpha}$ so that in the entire space, it is described by the same analytic expression as in the domain $r_{\alpha} \geq a_{\alpha}$. (For short, we call this extension an analytic extension.) The nonzero extension of the fluid pressure defined by (2.17) and the analytic extension of the external force field differ from the usually accepted zeroth extension of both the fluid pressure [19–22] and the external force field [23–26] to the domains $r_\alpha < a_\alpha$. It is worth noting that the considered problem can be solved for various extensions of the quantities $p(\boldsymbol{r}, t)$ and $\boldsymbol{F}^{ext}(\boldsymbol{r}, t)$ to the domains $r_\alpha < a_\alpha$ including the zeroth extension [19–26] and the extensions used in the present paper. However, the complexity of calculations and the possibility of interpretation of certain results essentially depend on the specific method of extension.

If the fluid pressure is extended to the domains $r_\alpha < a_\alpha$ according to relation (2.17), then the function $p(\boldsymbol{r}, t)$ defined in the entire space has a discontinuity at the surfaces $r_\alpha = a_\alpha$. For this reason, in what follows, we write the fluid pressure at the point $\boldsymbol{r} = \boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha$ as $p(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha + 0, t)$.

Note that for the analytic extension of the external force field $\boldsymbol{F}^{ext}(\boldsymbol{r}, t)$ to the domains $r_\alpha < a_\alpha$, the first two terms in (2.17) represent the pressure of the unbounded fluid without particles in the external force field $\boldsymbol{F}^{ext}(\boldsymbol{r}, t)$. In the case where the fluid velocity satisfies the stick boundary conditions at the surfaces of the spheres [1,4,5] (this case is considered in the present paper), condition (2.16) ensures the continuity of the function $\boldsymbol{v}(\boldsymbol{r}, t)$ given in the entire space at $r_\alpha = a_\alpha$.

In [24–26], the external force field $\boldsymbol{F}^{ext}(\boldsymbol{r}, t)$ is extended to the domains $r_\alpha < a_\alpha$ by zero and the stress tensor $\boldsymbol{P}(\boldsymbol{r}, t)$ in these domains is defined as follows:

$$\operatorname{div} \boldsymbol{P}(\boldsymbol{r}, t) = -\rho \frac{\partial \boldsymbol{U}_\alpha(\boldsymbol{r}, t)}{\partial t}, \quad r_\alpha < a_\alpha. \tag{2.18}$$

We note that in the case where the fluid velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ is extended to the domains $r_\alpha < a_\alpha$ according to relation (2.16) (in fact, this is used in [24–26]), the stress tensor $\boldsymbol{P}(\boldsymbol{r}, t)$ for $r_\alpha < a_\alpha$ must have another analytic representation than (2.3). Otherwise, if the stress tensor $\boldsymbol{P}(\boldsymbol{r}, t)$ in the domains $r_\alpha < a_\alpha$ is defined by relation (2.3), where the quantity $\boldsymbol{v}(\boldsymbol{r}, t)$ is defined by relation (2.16) in these domains, then for $r_\alpha < a_\alpha$, we have

$$\boldsymbol{P}(\boldsymbol{r}, t) = p(\boldsymbol{r}, t)\boldsymbol{I}, \tag{2.19}$$

where the quantity $p(\boldsymbol{r}, t)$ for $r_\alpha < a_\alpha$ depends on the method of extension of the fluid pressure $p(\boldsymbol{r}, t)$ defined for $r_\alpha \geq a_\alpha$ to $r_\alpha < a_\alpha$. However, for any extension of the fluid pressure $p(\boldsymbol{r},t)$ to the domains $r_\alpha < a_\alpha$, according to (2.19), the left-hand side of Eq. (2.18) contains the potential vector $\boldsymbol{\nabla} p(\boldsymbol{r},t)$, while the right-hand side of this equations contains the solenoidal vector -$\rho\frac{\partial}{\partial t}\left((\boldsymbol{\Omega}_\alpha(t)\times\boldsymbol{r})\right)$ because

$$(\boldsymbol{\Omega}_\alpha(t)\times\boldsymbol{r}) = \frac{1}{3}\,\mathbf{rot}\Big((\boldsymbol{\Omega}_\alpha(t)\times\boldsymbol{r})\times\boldsymbol{r}\Big). \tag{2.20}$$

This means that the extension of the fluid velocity $\boldsymbol{v}(\boldsymbol{r},t)$ to the domains $r_\alpha < a_\alpha$ according to (2.16) eliminates the possibility of extension (2.18) (used in [24–26]) for the stress tensor $\boldsymbol{P}(\boldsymbol{r},t)$ defined by relations (2.3) in the entire space to these domains for the case of the time-dependent angular velocity $\boldsymbol{\Omega}_\alpha(t)$.

The quantities $\boldsymbol{v}(\boldsymbol{r},t)$ and $p(\boldsymbol{r},t)$ extended to $r_\alpha < a_\alpha$ according to (2.16) and (2.17) and $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ analytically extended to these domains are given in the entire space and satisfy the continuity equation (2.2) and the equation

$$\rho\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t} + \operatorname{div}\boldsymbol{P}(\boldsymbol{r},t) = \boldsymbol{F}^{ext}(\boldsymbol{r},t) + \boldsymbol{F}^{ind}(\boldsymbol{r},t) \tag{2.21}$$

given in the entire space. Equation (2.21) differs from Eq. (2.1) by the additional term (the induced force density) $\boldsymbol{F}^{ind}(\boldsymbol{r},t)$ that appears due to the extension of the linearized Navier–Stokes equation given for $r_\alpha \geq a_\alpha$ to the domains $r_\alpha < a_\alpha$. For the analytic extension of the external force field, the quantity $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ in Eqs. (2.1) and (2.21) is described by the same analytic expressions, while, for the zeroth extension of the external force field to $r_\alpha < a_\alpha$ [19,23–26], the analytic expressions describing the quantity $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ in the domains $r_\alpha \geq a_\alpha$ and $r_\alpha < a_\alpha$ are different. Therefore, in the latter case, the procedure where the quantity $\boldsymbol{F}^{ind}(\boldsymbol{r},t)$ is formally considered to be equal to zero in Eq. (2.21) is not equivalent to the removal of the particles from the fluid because Eq. (2.1) given in the entire space and Eq. (2.21) with $\boldsymbol{F}^{ind}(\boldsymbol{r},t) = 0$ describe the unbounded fluid in various force fields, namely, in $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ and in $\sum_{\alpha=1}^{N}\Theta(r_\alpha - a_\alpha)\boldsymbol{F}^{ext}(\boldsymbol{r},t)$, where $\Theta(x)$ is the Heaviside function, respectively.

The induced force density $\boldsymbol{F}^{ind}(\boldsymbol{r},t)$ is presented in the form

$$\boldsymbol{F}^{ind}(\boldsymbol{r},t) = \sum_{\alpha=1}^{N}\boldsymbol{F}^{ind}_\alpha(\boldsymbol{r},t), \tag{2.22}$$

where $\boldsymbol{F}^{ind}_{\alpha}(\boldsymbol{r},t)$ is the induced force density for particle α. This quantity has the volume $\boldsymbol{F}^{(V)ind}_{\alpha}(\boldsymbol{r},t)$ and surface $\boldsymbol{F}^{(S)ind}_{\alpha}(\boldsymbol{r},t)$ components given, respectively, inside the volume $V_\alpha = (4/3)\pi a_\alpha^3$ occupied by the particle α and on its surface $S_\alpha(t)$ [22]

$$\boldsymbol{F}^{ind}_{\alpha}(\boldsymbol{r},t) = \boldsymbol{F}^{(V)ind}_{\alpha}(\boldsymbol{r},t) + \boldsymbol{F}^{(S)ind}_{\alpha}(\boldsymbol{r},t). \tag{2.23}$$

Here,

$$\boldsymbol{F}^{(V)ind}_{\alpha}(\boldsymbol{r},t) = \Theta(a_\alpha - r_\alpha)\left\{\rho\frac{\partial}{\partial t}\left(\boldsymbol{\Omega}_\alpha(t)\times\boldsymbol{r}_\alpha\right) - \boldsymbol{F}^{(sol)ext}(\boldsymbol{r},t)\right\}, \tag{2.24}$$

$$\boldsymbol{F}^{(S)ind}_{\alpha}(\boldsymbol{r},t) = \int d\Omega_\alpha\, \delta(\boldsymbol{r} - \boldsymbol{R}_\alpha(t) - \boldsymbol{a}_\alpha)\boldsymbol{f}_\alpha(\boldsymbol{a}_\alpha,t), \tag{2.25}$$

where $\boldsymbol{a}_\alpha \equiv (a_\alpha, \theta_\alpha, \varphi_\alpha)$ is the vector directed from the center of sphere α to a point on its surface characterized by the polar θ_α and azimuth φ_α angles in the local spherical coordinate system O_α, $d\Omega_\alpha = \sin\theta_\alpha\, d\theta_\alpha d\varphi_\alpha$ is the solid angle, and $\boldsymbol{f}_\alpha(\boldsymbol{a}_\alpha,t)$ is a certain unknown density of the induced surface force distributed over the surface $S_\alpha(t)$.

Unlike [21], the volume component $\boldsymbol{F}^{(V)ind}_{\alpha}(\boldsymbol{r},t)$ of the induced force does not contain the translational velocity $\boldsymbol{U}_\alpha(t)$ of particle α, which is caused by the different extensions of the fluid pressure to $r_\alpha < a_\alpha$ used in the present paper and in [21]. In addition, owing to extension (2.17) for the fluid pressure to $r_\alpha < a_\alpha$, $\boldsymbol{F}^{(V)ind}_{\alpha}(\boldsymbol{r},t)$ is defined only by the solenoidal component of the analytically extended external force field. Thus, in the particular case of conservative external forces, $\boldsymbol{F}^{(V)ind}_{\alpha}(\boldsymbol{r},t)$ is independent of external forces.

It is easy to verify that the representation of the induced force density in the form (2.22)–(2.25) as well as relations (2.16) and (2.17) and the analytic extension of $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ to $r_\alpha < a_\alpha$ insure the validity of Eq. (2.21) in the entire space. In this case,

$$\operatorname{div}\boldsymbol{P}(\boldsymbol{r},t) = -\rho\frac{d\boldsymbol{U}_\alpha(t)}{dt} + \boldsymbol{F}^{(p)ext}(\boldsymbol{r},t) + \boldsymbol{F}^{(S)ind}_{\alpha}(\boldsymbol{r},t), \quad r_\alpha \le a_\alpha. \tag{2.26}$$

Using relation (2.26), we can represent force (2.14) and torque (2.15) exerted by the fluid on sphere α as follows:

$$\boldsymbol{F}^{f}_{\alpha}(t) = \boldsymbol{F}_\alpha(t) + \boldsymbol{F}^{in}_{\alpha}(t) - \tilde{\boldsymbol{F}}^{(p)ext}_{\alpha}(t), \tag{2.27}$$

$$\boldsymbol{T}^{f}_{\alpha}(t) = \boldsymbol{T}_\alpha(t) - \tilde{\boldsymbol{T}}^{(p)ext}_{\alpha}(t), \tag{2.28}$$

where $\boldsymbol{F}_\alpha(t)$ and $\boldsymbol{T}_\alpha(t)$ are, respectively, the force and the torque exerted on the sphere α due to the induced force density distributed over its surface

$$\boldsymbol{F}_\alpha(t) = -\int d\boldsymbol{r}\, \boldsymbol{F}_\alpha^{(S)ind}(\boldsymbol{r},t) = -\int d\Omega_\alpha\, \boldsymbol{f}_\alpha(\boldsymbol{a}_\alpha,t), \tag{2.29}$$

$$\boldsymbol{T}_\alpha(t) = -\int d\boldsymbol{r}\, \Big(\boldsymbol{r}_\alpha \times \boldsymbol{F}_\alpha^{(S)ind}(\boldsymbol{r},t)\Big) = -\int d\Omega_\alpha\, \Big(\boldsymbol{a}_\alpha \times \boldsymbol{f}_\alpha(\boldsymbol{a}_\alpha,t)\Big), \tag{2.30}$$

$\tilde{\boldsymbol{F}}_\alpha^{(p)ext}(t)$ and $\tilde{\boldsymbol{T}}_\alpha^{(p)ext}(t)$ are, respectively, the force and the torque acting on the fluid sphere occupying the volume V_α instead of spherical particle α due to the potential component $\boldsymbol{F}^{(p)ext}(\boldsymbol{r},t)$ of the external force field analytically extended to this domain

$$\tilde{\boldsymbol{F}}_\alpha^{(p)ext}(t) = \int_{V_\alpha} d\boldsymbol{r}\, \boldsymbol{F}^{(p)ext}(\boldsymbol{r},t), \tag{2.31}$$

$$\tilde{\boldsymbol{T}}_\alpha^{(p)ext}(t) = \int_{V_\alpha} d\boldsymbol{r}\, \Big(\boldsymbol{r}_\alpha \times \boldsymbol{F}^{(p)ext}(\boldsymbol{r},t)\Big), \tag{2.32}$$

and

$$\boldsymbol{F}_\alpha^{in}(t) = \tilde{m}_\alpha \frac{d\boldsymbol{U}_\alpha(t)}{dt} \tag{2.33}$$

is the inertial force that is necessary to be applied to a fluid sphere of volume V_α in order that it move with the acceleration $d\boldsymbol{U}_\alpha(t)/dt$ [19], where $\tilde{m}_\alpha = \rho V_\alpha$ is the mass of the fluid displaced by particle α.

In the particular case of the time-independent homogeneous gravitational force field

$$\varphi(\boldsymbol{r},t) \equiv \varphi(\boldsymbol{r}) = -\boldsymbol{g}\boldsymbol{r}, \qquad \boldsymbol{F}^{(sol)ext}(\boldsymbol{r},t) = 0, \tag{2.34}$$

we have

$$\begin{aligned} \boldsymbol{F}_\alpha^{ext}(t) &\equiv \boldsymbol{F}_\alpha^{ext} = m_\alpha \boldsymbol{g}, \qquad & \tilde{\boldsymbol{F}}_\alpha^{(p)ext}(t) &= -\boldsymbol{F}_\alpha^{A}, \\ \boldsymbol{T}_\alpha^{ext}(t) &= 0, & \tilde{\boldsymbol{T}}_\alpha^{(p)ext}(t) &= 0, \end{aligned} \tag{2.35}$$

where $\boldsymbol{F}_\alpha^A = -\tilde{m}_\alpha \boldsymbol{g}$ is the Archimedes force acting on particle α and $\boldsymbol{g}$ is the acceleration of gravity. Therefore, in the considered case, $\boldsymbol{F}_\alpha^{ext}(t) - \tilde{\boldsymbol{F}}_\alpha^{ext}(t) = (m_\alpha - \tilde{m}_\alpha)\,\boldsymbol{g}$ in relation (2.12) is the gravity force of particle α corrected for the buoyancy force.

In view of relations (2.29) and (2.30), the forces and torques exerted by the fluid on particles are completely determined by the surface induced force densities. If it is necessary to determine only the forces $\boldsymbol{F}_\alpha(t)$ and the torques $\boldsymbol{T}_\alpha(t)$ in the linear approximation with respect to the velocities of the fluid and the spheres, it is convenient to use the procedure proposed in [20–22], which does not require the explicit form of $\boldsymbol{F}^{(S)ind}(\boldsymbol{r}, t)$. On the basis of this procedure, for an arbitrary number of spheres in a fluid, one obtains the mobility tensors of particles that move and rotate in the fluid both in the stationary case [20,21] and with regard for the time dependence [22]. In a more general case where the velocity field of the fluid should be determined, the problem becomes essentially more complicated because the explicit form of the induced surface force densities must be obtained. In the nonstationary case, this problem is considered in [25,26] where the zeroth extension of the external forces to the domains occupied by spheres is used and the stress tensor in these domains is defined by relation (2.18).

III. REPRESENTATION OF THE REQUIRED QUANTITIES IN TERMS OF HARMONICS OF INDUCED SURFACE FORCE DENSITIES

To determine the velocity and pressure fields of the fluid in the presence of N spheres in it, we note the same structure of Eqs. (2.1) and (2.21). Therefore, we can use solution (2.7), (2.8) of Eqs. (2.1)–(2.3) with the substitution $\boldsymbol{F}^{ext}(\boldsymbol{k}, \omega) + \boldsymbol{F}^{ind}(\boldsymbol{k}, \omega)$ for $\boldsymbol{F}^{ext}(\boldsymbol{k}, \omega)$.

We consider the problem linearized both with respect to the velocity of the fluid and the velocities of the spheres. Within the framework of this approximation, we neglect the time dependence of the positions of centers of the spheres defined by $\boldsymbol{R}_\alpha(t)$ and their surfaces $S_\alpha(t)$. This means that the spheres do not displace considerably for considered time intervals (for details, see [22]). This enables us to represent $\boldsymbol{F}_\alpha^{(S)ind}(\boldsymbol{k}, \omega)$ and $\boldsymbol{F}_\alpha^{(V)ind}(\boldsymbol{k}, \omega)$ as follows:

$$\boldsymbol{F}_\alpha^{(S)ind}(\boldsymbol{k}, \omega) = \int d\Omega_\alpha \, \exp\left(-i\boldsymbol{k} \cdot (\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha)\right) \boldsymbol{f}_\alpha(\boldsymbol{a}_\alpha, \omega), \tag{3.1}$$

$$\boldsymbol{F}_\alpha^{(V)ind}(\boldsymbol{k}, \omega) = -\left\{ i\frac{2}{3}\xi_\alpha b_\alpha^2 \frac{j_2(ka_\alpha)}{k^2}\Big(\boldsymbol{\Omega}_\alpha(\omega) \times \boldsymbol{k}\Big) + \tilde{\boldsymbol{F}}_\alpha^{(sol)ext}(\boldsymbol{k}, \omega)\right\} \exp\left(-i\boldsymbol{k} \cdot \boldsymbol{R}_\alpha\right), \tag{3.2}$$

where $\xi_\alpha = 6\pi\eta a_\alpha$ is the Stokes friction coefficient for a sphere of radius a uniformly moving in the fluid, $b_\alpha = \kappa a_\alpha$ is the dimensionless parameter that characterizes the ratio of the radius a of sphere α to the length δ $[|b_\alpha| = \sqrt{2}\, a_\alpha/\delta)]$, $j_n(x)$ is the spherical Bessel function of order n, and

$$\tilde{\boldsymbol{F}}_\alpha^{(sol)ext}(\boldsymbol{k},\omega) = \int d\boldsymbol{r}_\alpha \, \exp\left(-i\boldsymbol{k}\cdot\boldsymbol{r}_\alpha\right) \Theta(a_\alpha - r_\alpha)\, \boldsymbol{F}^{(sol)ext}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha,\omega). \tag{3.3}$$

Note that the quantity $\tilde{\boldsymbol{F}}_\alpha^{(sol)ext}(\boldsymbol{k},\omega)$ defined by relation (3.3) differs from the Fourier transform $\boldsymbol{F}_\alpha^{(sol)ext}(\boldsymbol{k},\omega)$ of the solenoidal component of the external force field because the integral in (3.3) is taken over the finite space domain $r_\alpha \leq a_\alpha$, while for $\boldsymbol{F}_\alpha^{(sol)ext}(\boldsymbol{k},\omega)$, the Fourier integral is taken over the entire space.

By using relations (2.7), (2.8), and (3.1)–(3.3), we can represent the required Fourier transforms for the quantities $\boldsymbol{v}(\boldsymbol{r},\omega)$ and $p(\boldsymbol{r},\omega)$ defined in the entire space as follows:

$$\boldsymbol{v}(\boldsymbol{r},\omega) = \boldsymbol{v}^{(0)}(\boldsymbol{r},\omega) + \boldsymbol{v}^{ind}(\boldsymbol{r},\omega), \tag{3.4}$$

$$p(\boldsymbol{r},\omega) = p^{(0)}(\boldsymbol{r},\omega) + p^{ind}(\boldsymbol{r},\omega). \tag{3.5}$$

Here, the quantities $p^{(0)}(\boldsymbol{r},\omega)$ and $\boldsymbol{v}^{(0)}(\boldsymbol{r},\omega)$, where

$$\boldsymbol{v}^{(0)}(\boldsymbol{r},\omega) = \boldsymbol{v}^{(0)sol}(\boldsymbol{r},\omega) + \boldsymbol{v}^{inf}(\omega), \tag{3.6}$$

$$\boldsymbol{v}^{(0)sol}(\boldsymbol{r},\omega) = \frac{1}{(2\pi)^3\eta} \int\limits_{-\infty}^{\infty} d\boldsymbol{k}\, \frac{\exp\left(i\boldsymbol{k}\cdot\boldsymbol{r}\right)}{k^2+\kappa^2}\, \boldsymbol{F}^{(sol)ext}(\boldsymbol{k},\omega), \tag{3.7}$$

defined at any point of the space, for $r_\alpha \geq a_\alpha$, are, respectively, the pressure and the velocity of the unbounded fluid in the absence of particles defined by relations (2.8) and (3.6)–(3.7). The quantities $\boldsymbol{v}^{ind}(\boldsymbol{r},\omega)$ and $p^{ind}(\boldsymbol{r},\omega)$ can be represented in the form

$$\boldsymbol{v}^{ind}(\boldsymbol{r},\omega) = \sum_{\beta=1}^{N} \boldsymbol{v}_\beta^{ind}(\boldsymbol{r},\omega), \tag{3.8}$$

$$p^{ind}(\boldsymbol{r},\omega) = \sum_{\beta=1}^{N} p_\beta^{ind}(\boldsymbol{r},\omega), \tag{3.9}$$

where

$$\boldsymbol{v}_\beta^{ind}(\boldsymbol{r},\omega) = \boldsymbol{v}_\beta^{(V)ind}(\boldsymbol{r},\omega) + \boldsymbol{v}_\beta^{(S)ind}(\boldsymbol{r},\omega), \tag{3.10}$$

$$p_\beta^{ind}(\boldsymbol{r},\omega) = p_\beta^{(V)ind}(\boldsymbol{r},\omega) + p_\beta^{(S)ind}(\boldsymbol{r},\omega). \tag{3.11}$$

For $r_\alpha > a_\alpha$, $\boldsymbol{v}^{ind}(\boldsymbol{r},\omega)$ and $p^{ind}(\boldsymbol{r},\omega)$ are, respectively, the fluid velocity and pressure induced by all particles. Here,

$$\boldsymbol{v}_\beta^{(S)ind}(\boldsymbol{r},\omega) = \frac{1}{(2\pi)^3}\int\limits_{-\infty}^{\infty} d\boldsymbol{k}\, \exp\left(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_\beta)\right)\int d\Omega_\beta\, \exp\left(-i\boldsymbol{k}\boldsymbol{a}_\beta\right)\boldsymbol{S}(\boldsymbol{k},\omega)\cdot\boldsymbol{f}_\beta(\boldsymbol{a}_\beta,\omega), \tag{3.12}$$

$$p_\beta^{(S)ind}(\boldsymbol{r},\omega) = -\frac{i}{(2\pi)^3}\int\limits_{-\infty}^{\infty} d\boldsymbol{k}\,\frac{\exp\left(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_\beta)\right)}{k^2}\int d\Omega_\beta\, \exp\left(-i\boldsymbol{k}\boldsymbol{a}_\beta\right)\boldsymbol{k}\cdot\boldsymbol{f}_\beta(\boldsymbol{a}_\beta,\omega). \tag{3.13}$$

Relations (3.12) and (3.13) are valid for any $\boldsymbol{r}$. For $r_\alpha > a_\alpha$, the quantities $\boldsymbol{v}_\beta^{(S)ind}(\boldsymbol{r},\omega)$ and $p_\beta^{(S)ind}(\boldsymbol{r},\omega)$ can be interpreted, respectively, as the fluid velocity and pressure at the point $\boldsymbol{r}$ generated by the induced surface force $\boldsymbol{F}_\beta^{(S)ind}(\boldsymbol{r},\omega)$ distributed over the surface of sphere β. The quantity

$$\boldsymbol{S}(\boldsymbol{k},\omega) = \frac{1}{\eta\left(k^2+\kappa^2\right)}\left(\boldsymbol{I}-\boldsymbol{n}_k\boldsymbol{n}_k\right) \tag{3.14}$$

is the dynamic Oseen tensor.

The quantities $\boldsymbol{v}_\beta^{(V)ind}(\boldsymbol{r},\omega)$ and $p_\beta^{(V)ind}(\boldsymbol{r},\omega)$ defined in the entire space, in the domain $r_\alpha > a_\alpha$, are, respectively, the fluid velocity and pressure at the point $\boldsymbol{r}$ generated by the induced volume force $\boldsymbol{F}_\beta^{(V)ind}(\boldsymbol{r},\omega)$ given in the volume V_β occupied by particle β, moreover,

$$p_\beta^{(V)ind}(\boldsymbol{r},\omega) = \frac{i}{(2\pi)^3}\int\limits_{-\infty}^{\infty} d\boldsymbol{k}\,\frac{\exp\left(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_\beta)\right)}{k^2}\,\boldsymbol{k}\cdot\tilde{\boldsymbol{F}}_\beta^{(sol)ext}(\boldsymbol{k},\omega). \tag{3.15}$$

Thus, the contribution to the fluid pressure due to the volume induced forces is nonzero only if the considered system is in a nonconcervative force field.

It is convenient to represent the quantity $\boldsymbol{v}_\beta^{(V)ind}(\boldsymbol{r},\omega)$ as the sum of two terms

$$\boldsymbol{v}_\beta^{(V)ind}(\boldsymbol{r},\omega) = \boldsymbol{v}_\beta^{(r)ind}(\boldsymbol{r},\omega) + \boldsymbol{v}_\beta^{(sol)ind}(\boldsymbol{r},\omega), \tag{3.16}$$

where the first term is caused by the rotation of particle β

$$\boldsymbol{v}_\beta^{(r)ind}(\boldsymbol{r},\omega) = -i\xi_\beta b_\beta^2\frac{2}{3\eta}\frac{1}{(2\pi)^3}\int\limits_{-\infty}^{\infty} d\boldsymbol{k}\,\frac{\exp\left(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_\beta)\right)}{k^2+\kappa^2}\frac{j_2(\kappa a_\beta)}{k^2}\Big(\boldsymbol{\Omega}_\beta(\omega)\times\boldsymbol{k}\Big) \tag{3.17}$$

and the second is caused by the solenoidal component of the external force filed

$$\boldsymbol{v}_\beta^{(sol)ind}(\boldsymbol{r},\omega) = -\frac{1}{(2\pi)^3\eta}\int\limits_{-\infty}^{\infty} d\boldsymbol{k}\frac{\exp\left(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_\beta)\right)}{k^2+\kappa^2}\left(\boldsymbol{I}-\boldsymbol{n}_k\boldsymbol{n}_k\right)\cdot\tilde{\boldsymbol{F}}_\beta^{(sol)ext}(\boldsymbol{k},\omega). \tag{3.18}$$

Note that for the zeroth extension of the external forces to the domains $r_\alpha < a_\alpha$ used in [20,23–26], the interpretation of $\boldsymbol{v}^{(0)}(\boldsymbol{r},\omega)$ as the nonperturbed velocity of the fluid, i.e., the fluid velocity in the absence of the spheres, is not a quite correct because, in this case, $\boldsymbol{v}^{(0)}(\boldsymbol{r},\omega)$ is the velocity of the fluid not containing particles but in the external force field $\sum\limits_{\alpha=1}^{N}\Theta(r_\alpha - a_\alpha)\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ instead of $\boldsymbol{F}^{ext}(\boldsymbol{r},t)$ given in the entire space. The same remarks are also valid for the interpretation of $p^{(0)}(\boldsymbol{r},\omega)$.

We expand the induced surface force densities $\boldsymbol{f}_\beta(\boldsymbol{a}_\beta,\omega)$ in the series in the spherical harmonics

$$\boldsymbol{f}_\beta(\boldsymbol{a}_\beta,\omega) = \frac{1}{2\sqrt{\pi}}\sum_{lm}\boldsymbol{f}_{\beta,lm}(a_\beta,\omega)Y_{lm}(\theta_\beta,\varphi_\beta), \tag{3.19}$$

where, for short, the symbol $\sum\limits_{lm}$ means $\sum\limits_{l=0}^{\infty}\sum\limits_{m=-l}^{l}$ and the arguments a_β in $\boldsymbol{f}_{\beta,lm}(a_\beta,\omega)$ is omitted in what follows, and the spherical harmonics $Y_{lm}(\theta,\varphi)$ are defined as follows [31]:

$$\begin{aligned} Y_{lm}(\theta,\varphi) &= \frac{\exp\left(im\varphi\right)}{\sqrt{2\pi}}\Theta_{lm}(\cos\theta), \qquad l=0,1,\ldots, \qquad -l\le m\le l, \\ \Theta_{lm}(x) &= (-1)^{\frac{m-|m|}{2}}\sqrt{\frac{2l+1}{2}\frac{(l-|m|)!}{(l+|m|)!}}\,P_l^{|m|}(x), \end{aligned} \tag{3.20}$$

and $P_l^m(x)$ is the associated Legendre polynomial.

By using expansion (3.19) and relations (2.29) and (2.30), we can represent the Fourier transforms of the force $\boldsymbol{F}_\alpha(\omega)$ and the torque $\boldsymbol{T}_\alpha(\omega)$ exerted by the fluid on particle α in terms of Fourier harmonics of the induced surface force density $\boldsymbol{f}_{\alpha,lm}(\omega)$ as follows:

$$\boldsymbol{F}_\alpha(\omega) = -\boldsymbol{f}_{\alpha,00}(\omega), \tag{3.21}$$

$$\boldsymbol{T}_\alpha(\omega) = -\frac{a_\alpha}{\sqrt{3}}\sum_{m=-1}^{1}\left(\boldsymbol{e}_m\times\boldsymbol{f}_{\alpha,1m}(\omega)\right), \tag{3.22}$$

where

$$\boldsymbol{e}_0 \equiv \boldsymbol{e}_z, \qquad \boldsymbol{e}_{\pm1} = \frac{1}{\sqrt{2}}\left(i\boldsymbol{e}_y \pm \boldsymbol{e}_x\right), \tag{3.23}$$

i.e., up to the factor $(-1)^m$, where $m = 0, \pm 1$, are the cyclic covariant unit vectors [32], and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ are the Cartesian unit vectors.

Thus, the problem of determination of the forces and torques exerted by the fluid on particles is reduced to the determination of only the zeroth and first (with respect to l) harmonics of the induced surface force densities [25,26,30].

For any point of the space, its position defined by the radius vector $\boldsymbol{r}$ relative to the fixed system O can be represented in the form $\boldsymbol{r} = \boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha$, where $\boldsymbol{R}_\alpha$ is the radius vector that defines the position of the center of certain sphere α and $\boldsymbol{r}_\alpha$ is the radius vector directed from this center to the point at hand. Since there are N particles, there exist N different representations for $\boldsymbol{r}$. Among these representations, in the case where the point of observation belongs to the domain occupied by the fluid, it is convenient to take $\boldsymbol{r}_\alpha$ corresponding to the minimum difference $r_\beta - a_\beta$, where $\beta = 1, 2, \ldots, N$, which means the minimum distance from the point at hand to the surface of sphere α. If the point of observation lies inside the domain occupied by sphere α, then $\boldsymbol{r}_\alpha$ is the radius vector directed from the center of this sphere to this point.

In this case, for $r_\alpha > a_\alpha$, the quantities $\boldsymbol{v}_\beta^{(S)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha, \omega)$, $p_\beta^{(S)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha, \omega)$ and $\boldsymbol{v}_\beta^{(V)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha, \omega)$, $p_\beta^{(V)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha, \omega)$, where $\beta = 1, 2, \ldots, N$, are the velocities and the pressures of the fluid at the point $\boldsymbol{r} = \boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha$ induced, respectively, by the surface and volume forces distributed over the surface and inside the volume of particle β with the center defined by the vector $\boldsymbol{R}_\beta$. The particular case $\beta = \alpha$ corresponds to the velocities and the pressures of the fluid generated at this point, respectively, by the surface and volume forces distributed over the surface and inside the volume of the particle that is the most closely situated to this point. This enables us to separate contributions to the induced velocity and pressure of the fluid caused by the closest particle and more distant ones.

At any point $\boldsymbol{r}$ of the space, we can represent the quantities defined by relations (3.12), (3.13), (3.15), (3.17), and (3.18) as expansions in the spherical harmonics

$$\boldsymbol{v}_\beta^{(S)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha, \omega) = 2\sqrt{\pi} \sum_{lm} \boldsymbol{v}_{\beta,lm}^{(S)ind}(\boldsymbol{R}_\alpha, r_\alpha, \omega) Y_{lm}(\theta_\alpha, \varphi_\alpha), \tag{3.24}$$

$$p_{\beta}^{(S)ind}(\boldsymbol{R}_{\alpha}+\boldsymbol{r}_{\alpha},\omega) = 2\sqrt{\pi}\sum_{lm} p_{\beta,lm}^{(S)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega)Y_{lm}(\theta_{\alpha},\varphi_{\alpha}), \tag{3.25}$$

$$\boldsymbol{v}_{\beta}^{(r)ind}(\boldsymbol{R}_{\alpha}+\boldsymbol{r}_{\alpha},\omega) = 2\sqrt{\pi}\sum_{lm} \boldsymbol{v}_{\beta,lm}^{(r)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega)Y_{lm}(\theta_{\alpha},\varphi_{\alpha}), \tag{3.26}$$

$$\boldsymbol{v}_{\beta}^{(sol)ind}(\boldsymbol{R}_{\alpha}+\boldsymbol{r}_{\alpha},\omega) = 2\sqrt{\pi}\sum_{lm} \boldsymbol{v}_{\beta,lm}^{(sol)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega)Y_{lm}(\theta_{\alpha},\varphi_{\alpha}), \tag{3.27}$$

$$p_{\beta}^{(V)ind}(\boldsymbol{R}_{\alpha}+\boldsymbol{r}_{\alpha},\omega) = 2\sqrt{\pi}\sum_{lm} p_{\beta,lm}^{(V)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega)Y_{lm}(\theta_{\alpha},\varphi_{\alpha}), \tag{3.28}$$

where $\boldsymbol{r}_{\alpha} \equiv (r_{\alpha},\theta_{\alpha},\varphi_{\alpha})$ is the radius vector from the center of the particle closets to the considered point if $r_{\alpha} \geq a_{\alpha}$ or the radius vector from the center of the particle to the point of observation lying inside this particle if $r_{\alpha} < a_{\alpha}$ and θ_{α} and φ_{α} are, respectively, the polar and azimuth angles of this vector in the local spherical coordinate system O_{α}. The expansion coefficients in series (3.24)–(3.28) of the corresponding quantities in the spherical harmonics $Y_{lm}(\theta_{\alpha},\varphi_{\alpha})$ can be presented in the following form: For $\beta \neq \alpha$, we have

$$\boldsymbol{v}_{\beta,l_1m_1}^{(S)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega) = \sum_{l_2m_2} \boldsymbol{T}_{l_1m_1}^{l_2m_2}(r_{\alpha},a_{\beta},\boldsymbol{R}_{\alpha\beta},\omega)\cdot \boldsymbol{f}_{\beta,l_2m_2}(\omega), \tag{3.29}$$

$$p_{\beta,l_1m_1}^{(S)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega) = \sum_{l_2m_2} \boldsymbol{D}_{l_1m_1}^{l_2m_2}(r_{\alpha},a_{\beta},\boldsymbol{R}_{\alpha\beta})\cdot \boldsymbol{f}_{\beta,l_2m_2}(\omega), \tag{3.30}$$

$$\begin{aligned}\boldsymbol{v}_{\beta,l_1m_1}^{(r)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega) &= \frac{2}{3}\xi_{\beta}b_{\beta}^{2}\sum_{l_2m_2}(-1)^{l_2}P_{l_12,l_2}(r_{\alpha},a_{\beta},R_{\alpha\beta},\omega)\left(\boldsymbol{\Omega}_{\beta}(\omega)\times \boldsymbol{W}_{l_1m_1,00}^{l_2m_2}\right)\\ &\times Y_{l_2m_2}^{*}(\Theta_{\alpha\beta},\Phi_{\alpha\beta}),\end{aligned} \tag{3.31}$$

$$\begin{aligned}\boldsymbol{v}_{\beta,l_1m_1}^{(sol)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega) &= -2\sqrt{\pi}\int d\boldsymbol{r}_{\beta}\Theta(a_{\beta}-r_{\beta})\sum_{l_2m_2}\boldsymbol{T}_{l_1m_1}^{l_2m_2}(r_{\alpha},r_{\beta},\boldsymbol{R}_{\alpha\beta},\omega)\\ &\cdot\ \boldsymbol{F}^{(sol)ext}(\boldsymbol{R}_{\beta}+\boldsymbol{r}_{\beta},\omega)Y_{l_2m_2}^{*}(\theta_{\beta},\varphi_{\beta}),\end{aligned} \tag{3.32}$$

$$\begin{aligned}p_{\beta,l_1m_1}^{(V)ind}(\boldsymbol{R}_{\alpha},r_{\alpha},\omega) &= 2\sqrt{\pi}\int d\boldsymbol{r}_{\beta}\Theta(a_{\beta}-r_{\beta})\sum_{l_2m_2}\boldsymbol{D}_{l_1m_1}^{l_2m_2}(r_{\alpha},r_{\beta},\boldsymbol{R}_{\alpha\beta})\\ &\cdot\ \boldsymbol{F}^{(sol)ext}(\boldsymbol{R}_{\beta}+\boldsymbol{r}_{\beta},\omega)Y_{l_2m_2}^{*}(\theta_{\beta},\varphi_{\beta}),\end{aligned} \tag{3.33}$$

where $\boldsymbol{R}_{\alpha\beta} = \boldsymbol{R}_{\alpha} - \boldsymbol{R}_{\beta} \equiv (R_{\alpha\beta},\Theta_{\alpha\beta},\Phi_{\alpha\beta})$ is the vector between the centers of spheres α and β pointing from sphere β to sphere α in the spherical coordinate system O,

$$\boldsymbol{T}_{l_1m_1}^{l_2m_2}(r_{\alpha},r_{\beta},\boldsymbol{R}_{\alpha\beta},\omega) = \sum_{lm}\boldsymbol{T}_{l_1m_1,lm}^{l_2m_2}(r_{\alpha},r_{\beta},R_{\alpha\beta},\omega)Y_{lm}(\Theta_{\alpha\beta},\Phi_{\alpha\beta}), \tag{3.34}$$

$$\boldsymbol{T}_{l_1m_1,lm}^{l_2m_2}(r_{\alpha},r_{\beta},R_{\alpha\beta},\omega) = F_{l_1l_2,l}(r_{\alpha},r_{\beta},R_{\alpha\beta},\omega)\boldsymbol{K}_{l_1m_1,lm}^{l_2m_2}, \tag{3.35}$$

$$F_{l_1l_2,l}(r_{\alpha},r_{\beta},R_{\alpha\beta},\omega) = \frac{2}{\pi\eta}\int_{0}^{\infty}dk\,\frac{k^2}{k^2+\kappa^2}\,j_{l_1}(kr_{\alpha})j_{l_2}(kr_{\beta})j_{l}(kR_{\alpha\beta}), \tag{3.36}$$

$$\boldsymbol{K}^{l_2m_2}_{l_1m_1,lm} = i^{l_1-l_2+l} \int d\Omega_k \, (\boldsymbol{I} - \boldsymbol{n}_k\boldsymbol{n}_k) \, Y^*_{l_1m_1}(\theta_k,\varphi_k) Y_{l_2m_2}(\theta_k,\varphi_k) Y^*_{lm}(\theta_k,\varphi_k), \tag{3.37}$$

$$\boldsymbol{D}^{l_2m_2}_{l_1m_1}(r_\alpha, r_\beta, \boldsymbol{R}_{\alpha\beta}) = \sum_{lm} \boldsymbol{D}^{l_2m_2}_{l_1m_1,lm}(r_\alpha, r_\beta, R_{\alpha\beta}) Y_{lm}(\Theta_{\alpha\beta}, \Phi_{\alpha\beta}), \tag{3.38}$$

$$\boldsymbol{D}^{l_2m_2}_{l_1m_1,lm}(r_\alpha, r_\beta, R_{\alpha\beta}) = C_{l_1l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}) \boldsymbol{W}^{l_2m_2}_{l_1m_1,lm}, \tag{3.39}$$

$$C_{l_1l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}) = \frac{2}{\pi} \int_0^\infty dk \, k \, j_{l_1}(kr_\alpha) j_{l_2}(kr_\beta) j_l(kR_{\alpha\beta}), \tag{3.40}$$

$$\boldsymbol{W}^{l_2m_2}_{l_1m_1,lm} = i^{l_1-l_2+l-1} \int d\Omega_k \, \boldsymbol{n}_k \, Y^*_{l_1m_1}(\theta_k,\varphi_k) Y_{l_2m_2}(\theta_k,\varphi_k) Y^*_{lm}(\theta_k,\varphi_k), \tag{3.41}$$

$$P_{l_1l_2,l}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega) = \frac{2}{\pi\eta} \int_0^\infty dk \, \frac{k}{k^2+\kappa^2} \, j_{l_1}(kr_\alpha) j_{l_2}(ka_\beta) j_l(kR_{\alpha\beta}). \tag{3.42}$$

The quantities $\boldsymbol{K}^{l_2m_2}_{l_1m_1,lm}$ and $\boldsymbol{W}^{l_2m_2}_{l_1m_1,lm}$ defined by relations (3.37) and (3.41) can be represented in the explicit form in terms of the Wigner 3j-symbols [32,33]. (For the tensor $\boldsymbol{K}^{l_2m_2}_{l_1m_1,lm}$, this representation is given in [30].) The corresponding relations have the form

$$\begin{aligned}
\boldsymbol{K}^{l_2m_2}_{l_1m_1,lm} &= \frac{i^{l_1-l_2+l}}{3\sqrt{\pi}} \sqrt{(2l_1+1)(2l_2+1)(2l+1)} \, (-1)^{m_1+m} \left\{ \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ -m_1 & m_2 & m \end{pmatrix} \boldsymbol{I} \right. \\
&\quad - (-1)^m \frac{1}{2}\sqrt{\frac{3}{2}} \sum_{k=-2}^{2} (-1)^k \boldsymbol{K}_k \sum_{j=j_{min}}^{1} (2L+1) \begin{pmatrix} 2 & l & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \left. \times \begin{pmatrix} 2 & l & L \\ k & -m & m-k \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ -m_1 & m_2 & k-m \end{pmatrix} \right\}, \qquad l_1, l_2, l \geq 0,
\end{aligned} \tag{3.43}$$

where

$$L = l + 2j, \qquad j_{min} = \begin{cases} 1, & \text{if} \quad l = 0 \\ 0, & \text{if} \quad l = 1 \\ -1, & \text{if} \quad l \geq 2, \end{cases}$$

$$\begin{aligned}
\boldsymbol{K}_0 &= \sqrt{\frac{2}{3}} \left(-\boldsymbol{e}_x\boldsymbol{e}_x - \boldsymbol{e}_y\boldsymbol{e}_y + 2\boldsymbol{e}_z\boldsymbol{e}_z \right) = \sqrt{\frac{2}{3}} \left(\boldsymbol{e}_1\boldsymbol{e}_{-1} + \boldsymbol{e}_{-1}\boldsymbol{e}_1 + 2\boldsymbol{e}_0\boldsymbol{e}_0 \right), \\
\boldsymbol{K}_1 &= \boldsymbol{e}_x\boldsymbol{e}_z + \boldsymbol{e}_z\boldsymbol{e}_x - i\left(\boldsymbol{e}_y\boldsymbol{e}_z + \boldsymbol{e}_z\boldsymbol{e}_y \right) = -\sqrt{2} \left(\boldsymbol{e}_{-1}\boldsymbol{e}_0 + \boldsymbol{e}_0\boldsymbol{e}_{-1} \right), \\
\boldsymbol{K}_{-1} &= -\boldsymbol{K}_1^* = -\sqrt{2} \left(\boldsymbol{e}_1\boldsymbol{e}_0 + \boldsymbol{e}_0\boldsymbol{e}_1 \right), \\
\boldsymbol{K}_2 &= \boldsymbol{e}_x\boldsymbol{e}_x - \boldsymbol{e}_y\boldsymbol{e}_y - i\left(\boldsymbol{e}_x\boldsymbol{e}_y + \boldsymbol{e}_y\boldsymbol{e}_x \right) = 2\boldsymbol{e}_{-1}\boldsymbol{e}_{-1}, \\
\boldsymbol{K}_{-2} &= \boldsymbol{K}_2^* = 2\boldsymbol{e}_1\boldsymbol{e}_1,
\end{aligned} \tag{3.44}$$

and

$$\boldsymbol{W}^{l_2 m_2}_{l_1 m_1, lm} = \frac{i^{l_1-l_2+l-1}}{2\sqrt{\pi}} \sqrt{(2l_1+1)(2l_2+1)(2l+1)}\,(-1)^{m-m_2} \sum_{k=-1}^{1} \boldsymbol{e}^*_k \sum_{j=j_{min}}^{1} (2L+1)$$
$$\times \begin{pmatrix} 1 & l & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & l & L \\ k & -m & m-k \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ -m_1 & m_2 & k-m \end{pmatrix},$$
$$l_1, l_2, l \geq 0. \qquad (3.45)$$

where

$$L = l + 2j - 1, \qquad j_{min} = \begin{cases} 1, & \text{if } l = 0 \\ 0, & \text{if } l \geq 1. \end{cases}$$

Relation (3.43) coincides with relation (28) given in [30] up to the factor $(-1)^{(m_1-|m_1|+m_2-|m_2|+m-|m|)/2}$, which is caused by the different definition of the spherical harmonics $Y_{lm}(\theta,\varphi)$ for $m<0$ in the present paper and in [30].

In view of the properties of 3j-symbols, we have

$$\boldsymbol{K}^{l_2 m_2}_{l_1 m_1, lm} \neq 0 \qquad \text{only for} \qquad l = l_1 + l_2 - 2p, \qquad l \geq 0, \qquad (3.46)$$

where $p = -1, 0, 1, \ldots, p_{max}$, $p_{max} = \min\Big([(l_1+l_2)/2]\,, 1 + \min(l_1, l_2)\Big)$, $[a]$ is the integer part of a, and $\min(a,b)$ means the smallest quantity of a and b, and

$$\boldsymbol{W}^{l_2 m_2}_{l_1 m_1, lm} \neq 0 \qquad \text{only for} \qquad l = l_1 + l_2 - 2p + 1, \qquad l \geq 0, \qquad (3.47)$$

where $p = 0, 1, \ldots, \tilde{p}_{max}$, $\tilde{p}_{max} = \min\Big([(l_1+l_2+1)/2]\,, 1 + \min(l_1, l_2)\Big)$.

According to (3.46) and (3.47), the sums over l in relations (3.34) and (3.38) contain only terms with l satisfying these conditions. Therefore, it is necessary to investigate the quantities $F_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$ and $C_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta})$ only for the values of l given by (3.46) and (3.47), respectively. According to relations (3.29), (3.30) and (3.32), (3.33), the corresponding quantities in them are defined by general relations (3.34)–(3.41) for $r_\beta = a_\beta$ and $0 \leq r_\beta \leq a_\beta$, respectively.

In the particular case $l=0$, the general relations (3.43) and (3.45) for the quantities $\boldsymbol{K}^{l_2 m_2}_{l_1 m_1, lm}$ and $\boldsymbol{W}^{l_2 m_2}_{l_1 m_1, lm}$ are simplified to the form

$$\boldsymbol{K}^{l_2m_2}_{l_1m_1,00} = \frac{i^{l_1-l_2}}{2\sqrt{\pi}} \int d\Omega_k \, (\boldsymbol{I} - \boldsymbol{n}_k\boldsymbol{n}_k) \, Y^*_{l_1m_1}(\theta_k, \varphi_k) Y_{l_2m_2}(\theta_k, \varphi_k)$$

$$= \frac{1}{3\sqrt{\pi}} \left\{ \delta_{l_1,l_2} \, \delta_{m_1,m_2} \, \boldsymbol{I} - \sqrt{\frac{3(2l_1+1)(2l_2+2)}{8}} \, i^{l_1-l_2} (-1)^{m_1} \begin{pmatrix} l_1 & l_2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \right.$$

$$\left. \times \sum_{k=-2}^{2} \delta_{k,m_1-m_2} \begin{pmatrix} l_1 & l_2 & 2 \\ -m_1 & m_2 & k \end{pmatrix} \boldsymbol{K}_k \right\} \tag{3.48}$$

and

$$\boldsymbol{W}^{l_2m_2}_{l_1m_1,00} = \frac{i^{l_1-l_2-1}}{2\sqrt{\pi}} \int d\Omega_k \, \boldsymbol{n}_k \, Y^*_{l_1m_1}(\theta_k, \varphi_k) Y_{l_2m_2}(\theta_k, \varphi_k)$$

$$= \frac{i^{l_1-l_2-1}}{2\sqrt{\pi}} \sqrt{(2l_1+1)(2l_2+2)} (-1)^{m_1} \begin{pmatrix} l_1 & l_2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times \sum_{k=-1}^{1} \delta_{k,m_1-m_2} \begin{pmatrix} l_1 & l_2 & 1 \\ -m_1 & m_2 & k \end{pmatrix} \boldsymbol{e}^*_k. \tag{3.49}$$

Taking the properties of 3j-symbols into account, we obtain that

$$\boldsymbol{K}^{l_2m_2}_{l_1m_1,00} \neq 0 \quad \text{only for} \quad l_2 = l_1 + 2p \geq 0, \quad \text{where} \quad p = \begin{cases} 0, 1, & \text{if } \ l_1 = 0, 1 \\ 0, \pm 1, & \text{if } \ l_1 \geq 2, \end{cases} \tag{3.50}$$

and

$$\boldsymbol{W}^{l_2m_2}_{l_1m_1,00} \neq 0 \quad \text{only for} \quad l_2 = l_1 + 2p + 1 \geq 0, \quad \text{where} \quad p = \begin{cases} 0, & \text{if } \ l_1 = 0, 1 \\ 0, -1, & \text{if } \ l_1 \geq 1. \end{cases} \tag{3.51}$$

According to (3.51), the sum over l_2 in relation (3.31) contains only terms for $l_2 = l_1 \pm 1 \geq 0$. This means that the quantity $P_{l_1 2, l_2}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega)$ contained in this relation must be determined only for these values of l_2.

In the domains $r_\alpha \leq R_{\alpha\beta} - r_\beta$ and $r_\alpha \geq R_{\alpha\beta} + r_\beta$ (far from both particles), where, according to (3.32) and (3.33), $r_\beta \leq a_\beta$, the integrals in (3.36) and (3.40) that define the quantities $F_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$ and $C_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta})$ can be represented in the explicit form for the values of l given by conditions (3.46) and (3.47), respectively, (for details of calculation, see Appendix). For $r_\alpha \leq R_{\alpha\beta} - r_\beta$, where $0 \leq r_\beta \leq a_\beta$, we have

$$F_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega) = (-1)^p \frac{2\kappa}{\pi\eta} \left\{ \tilde{j}_{l_1}(x_\alpha) \tilde{j}_{l_2}(x_\beta) \tilde{h}_l(y_{\alpha\beta}) - \delta_{l,l_1+l_2+2} \frac{\pi^{3/2}}{2} \frac{\Gamma\left(l_1+l_2+\frac{5}{2}\right)}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)} \right.$$
$$\left. \times \frac{r_\alpha^{l_1} r_\beta^{l_2}}{y_{\alpha\beta}^3 R_{\alpha\beta}^{l_1+l_2}} \right\}, \quad l = l_1 + l_2 - 2p \geq 0, \quad p = -1, 0, 1, \ldots, p_{max}, \quad (3.52)$$

$$C_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}) = \delta_{l,l_1+l_2+1} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(l_1+l_2+\frac{3}{2}\right)}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)} \frac{r_\alpha^{l_1} r_\beta^{l_2}}{R_{\alpha\beta}^{l_1+l_2+2}},$$
$$l = l_1 + l_2 - 2p + 1 \geq 0, \qquad p = 0, 1, \ldots, \tilde{p}_{max}, \quad (3.53)$$

where $\tilde{j}_l(x) = \sqrt{\pi/(2x)} I_{l+\frac{1}{2}}(x)$ and $\tilde{h}_l(x) = \sqrt{\pi/(2x)} K_{l+\frac{1}{2}}(x)$ are the modified spherical Bessel functions of the first and third kind, respectively, [34], $\Gamma(z)$ is the gamma function, $x_\alpha = \kappa r_\alpha$, $x_\beta = \kappa r_\beta$, and $y_{\alpha\beta} = \kappa R_{\alpha\beta}$. The dimensionless parameter $y_{\alpha\beta} = \kappa R_{\alpha\beta}$ [$|y_{\alpha\beta}| = \sqrt{2}\, R_{\alpha\beta}/\delta$] characterizes the ratio of the distance between particles α and β to the depth of penetration δ of a plane transverse wave of frequency ω into the fluid. In a certain frequency range, the quantity $|y_{\alpha\beta}|$ can be both smaller (for short distances between particles) and greater (for space-apart particles) than one.

Note that relations (3.52) and (3.53) for $F_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$ and $C_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta})$ defining the induced fluid velocity and pressure were obtained without imposing any additional restrictions on the size of particles, distances between them, and the frequency range.

In the particular case of equal spheres ($a_\alpha \equiv a, \quad \alpha = 1, 2, \ldots, N$), if the observation point lies at the surface of a sphere ($r_\alpha = a$) and $r_\beta = a$, relation (3.52) for $l = l_1 + l_2 - 2p$, where $p = 0, 1, \ldots, p_{max}$ coincides with the corresponding relation (4.11) in [26] up to the factor $(-1)^p$. For $l = l_1 + l_2 + 2$, relation (3.52) agrees with (4.11) in [26] up to the factor $(l_1 + l_2 + 5/2)$ in the first and second terms in (4.11) and the omitted factor $1/\kappa!$ (in terms of the notation used in [26]) in the third term presented in (4.11) in [26] as the infinite series.

For $r_\alpha \geq R_{\alpha\beta} + r_\beta$, where $0 \leq r_\beta \leq a_\beta$, we have

$$F_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega) = (-1)^{l_2-p} \frac{2\kappa}{\pi\eta} \left\{ \tilde{h}_{l_1}(x_\alpha) \tilde{j}_{l_2}(x_\beta) \tilde{j}_l(y_{\alpha\beta}) - \delta_{l,l_1-l_2-2} \frac{\pi^{3/2}}{2} \right.$$
$$\left. \times \frac{\Gamma\left(l_1+\frac{1}{2}\right)}{\Gamma\left(l_2+\frac{3}{2}\right)\Gamma\left(l_1-l_2-\frac{1}{2}\right)} \frac{r_\beta^{l_2} R_{\alpha\beta}^l}{x_\alpha^3 r_\alpha^{l_1-2}} \right\},$$
$$l = l_1 + l_2 - 2p \geq 0, \qquad p = -1, 0, 1, \ldots, p_{max}, \quad (3.54)$$

$$C_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(l_1+\frac{1}{2}\right)}{\Gamma\left(l_2+\frac{3}{2}\right)} \frac{r_\beta^{l_2} R_{\alpha\beta}^{l_1-l_2-1}}{r_\alpha^{l_1+1}} \left\{ \delta_{l,l_1-l_2-1} \frac{1}{\Gamma\left(l_1-l_2+\frac{1}{2}\right)} + \delta_{l,l_1-l_2-3} \right.$$
$$\left. \times \frac{2}{\Gamma\left(l_1-l_2-\frac{3}{2}\right)} \left[\frac{1}{2l_1-1} \left(\frac{r_\alpha}{R_{\alpha\beta}}\right)^2 - \frac{1}{2l_2+3} \left(\frac{r_\beta}{R_{\alpha\beta}}\right)^2 - \frac{1}{2l+3} \right] \right\}$$
$$l = l_1 + l_2 - 2p + 1 \geq 0, \qquad p = 0, 1, \ldots, \tilde{p}_{max}. \quad (3.55)$$

Note that the quantities $F_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$ and $C_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta})$ defined by (3.54) and (3.55) for $r_\alpha \geq R_{\alpha\beta} + r_\beta$ can also be represented in the form (3.52) and (3.53), which are valid for $r_\alpha \leq R_{\alpha\beta} - r_\beta$, performing the changes $r_\alpha \leftrightarrow R_{\alpha\beta}$ and $l_1 \leftrightarrow l$ and putting, respectively, $l_1 = l + l_2 - 2s$, where $s = -1, 0, 1, \ldots, s_{max}$ and $s_{max} = \min\Big([(l+l_2)/2], 1 + \min(l, l_2)\Big)$, and $l_1 = l + l_2 + 1 - 2s$, where $s = 0, 1, \ldots, \tilde{s}_{max}$ and $\tilde{s}_{max} = \min\Big([(l+l_2+1)/2], 1 + \min(l, l_2)\Big)$.

The dimensionless parameter $\sigma_{\beta\alpha} = a_\beta / R_{\alpha\beta}$ (as well as $\sigma_{\alpha\beta} = a_\alpha / R_{\alpha\beta}$), which is the ratio of the radius of particles to the distance between them, is always smaller than one (for spheres of equal radii, it cannot be greater than 1/2). In diluted suspensions, $\sigma_{\beta\alpha} \ll 1$.

For $l = l_1 \pm 1 \geq 0$, the quantity $P_{l_1 2,l}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega)$ defined by relation (3.42) with $l_2 = 2$ can be represented as follows (see Appendix):

$$P_{l_1 2,l}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega) = (-1)^p \frac{2}{\pi\eta} \tilde{j}_{l_1}(x_\alpha) \tilde{j}_2(b_\beta) \tilde{h}_l(y_{\alpha\beta}) \tag{3.56}$$

for $r_\alpha \leq R_{\alpha\beta} - a_\beta$ and

$$P_{l_1 2,l}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega) = (-1)^{p+1} \frac{2}{\pi\eta} \tilde{h}_{l_1}(x_\alpha) \tilde{j}_2(b_\beta) \tilde{j}_l(y_{\alpha\beta}) \tag{3.57}$$

for $r_\alpha \geq R_{\alpha\beta} + a_\beta$. Here, $b_\beta = \kappa a_\beta$.

In the particular case $\beta = \alpha$, we have

$$\boldsymbol{v}^{(S)ind}_{\alpha,l_1 m_1}(\boldsymbol{R}_\alpha, r_\alpha, \omega) = \sum_{l_2 m_2} \boldsymbol{T}^{l_2 m_2}_{l_1 m_1}(r_\alpha, a_\alpha, \omega) \cdot \boldsymbol{f}_{\alpha, l_2 m_2}(\omega), \tag{3.58}$$
$$p^{(S)ind}_{\alpha,l_1 m_1}(\boldsymbol{R}_\alpha, r_\alpha, \omega) = \sum_{l_2 m_2} \boldsymbol{D}^{l_2 m_2}_{l_1 m_1}(r_\alpha, a_\alpha) \cdot \boldsymbol{f}_{\alpha, l_2 m_2}(\omega), \tag{3.59}$$
$$\boldsymbol{v}^{(r)ind}_{\alpha,l_1 m_1}(\boldsymbol{R}_\alpha, r_\alpha, \omega) = \delta_{l_1,1} \frac{\xi_\alpha b_\alpha^2}{6\pi\sqrt{3}} P_{12}(r_\alpha, a_\alpha, \omega) \Big(\boldsymbol{\Omega}_\alpha(\omega) \times \boldsymbol{e}^*_{m_1} \Big), \tag{3.60}$$
$$\boldsymbol{v}^{(sol)ind}_{\alpha,l_1 m_1}(\boldsymbol{R}_\alpha, r_\alpha, \omega) = -\int d\boldsymbol{r}'_\alpha \Theta(a_\alpha - r'_\alpha) \sum_{l_2 m_2} \boldsymbol{T}^{l_2 m_2}_{l_1 m_1}(r_\alpha, r'_\alpha, \omega)$$

$$\cdot\ \boldsymbol{F}^{(sol)ext}(\boldsymbol{R}_\alpha + \boldsymbol{r}'_\alpha, \omega) Y^*_{l_2 m_2}(\theta'_\alpha, \varphi'_\alpha), \tag{3.61}$$

$$p^{(V)ind}_{\alpha, l_1 m_1}(\boldsymbol{R}_\alpha, r_\alpha, \omega) = \int d\boldsymbol{r}'_\alpha \Theta(a_\alpha - r'_\alpha) \sum_{l_2 m_2} \boldsymbol{D}^{l_2 m_2}_{l_1 m_1}(r_\alpha, r'_\alpha)$$

$$\cdot\ \boldsymbol{F}^{(sol)ext}(\boldsymbol{R}_\alpha + \boldsymbol{r}'_\alpha, \omega) Y^*_{l_2 m_2}(\theta'_\alpha, \varphi'_\alpha), \tag{3.62}$$

where

$$\boldsymbol{T}^{l_2 m_2}_{l_1 m_1}(r_\alpha, r'_\alpha, \omega) = \frac{1}{2\sqrt{\pi}} F_{l_1 l_2}(r_\alpha, r'_\alpha, \omega) \boldsymbol{K}^{l_2 m_2}_{l_1 m_1, 00}, \tag{3.63}$$

$$F_{l_1 l_2}(r_\alpha, r'_\alpha, \omega) = \frac{2}{\pi\eta} \int_0^\infty dk\, \frac{k^2}{k^2 + \kappa^2}\, j_{l_1}(k r_\alpha) j_{l_2}(k r'_\alpha), \tag{3.64}$$

$$\boldsymbol{D}^{l_2 m_2}_{l_1 m_1}(r_\alpha, r'_\alpha) = \frac{1}{2\sqrt{\pi}} C_{l_1 l_2}(r_\alpha, r'_\alpha) \boldsymbol{W}^{l_2 m_2}_{l_1 m_1, 00}, \tag{3.65}$$

$$C_{l_1 l_2}(r_\alpha, r'_\alpha) = \frac{2}{\pi} \int_0^\infty dk\, k\, j_{l_1}(k r_\alpha) j_{l_2}(k r'_\alpha), \tag{3.66}$$

$$P_{l_1 l_2}(r_\alpha, a_\alpha, \omega) = \frac{2}{\pi\eta} \int_0^\infty dk\, \frac{k}{k^2 + \kappa^2}\, j_{l_1}(k r_\alpha) j_{l_2}(k a_\alpha). \tag{3.67}$$

Taking into account that according to (3.60) only one harmonic of $\boldsymbol{v}^{(r)ind}_{\alpha, l_1 m_1}(\boldsymbol{R}_\alpha, r_\alpha, \omega)$ is not equal to zero, we can represent this component as follows:

$$\boldsymbol{v}^{(r)ind}_\alpha(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha, \omega) = \frac{\xi_\alpha b_\alpha^2}{6\pi}\, P_{1,2}(r_\alpha, a_\alpha, \omega) \Big(\boldsymbol{\Omega}_\alpha(\omega) \times \boldsymbol{n}_\alpha\Big), \tag{3.68}$$

where $\boldsymbol{n}_\alpha = \boldsymbol{r}_\alpha / r_\alpha$.

According to (3.50) and (3.51), the quantities $\boldsymbol{K}^{l_2 m_2}_{l_1 m_1, 00}$ and $\boldsymbol{W}^{l_2 m_2}_{l_1 m_1, 00}$ are nonzero only for at most three and two values of l_2, respectively. Therefore, the infinite sums $\sum\limits_{l_2=0}^{\infty}$ in relations (3.58), (3.61) and (3.59), (3.62) are replaced by sums containing, respectively, at most three and two terms corresponding to different values of l_2. This means that the quantities $F_{l_1 l_2}(r_\alpha, r'_\alpha, \omega)$ and $C_{l_1 l_2}(r_\alpha, r'_\alpha)$ should be determined only for these values of l_2.

Finally, for $r_\alpha \geq a_\alpha$ and $0 \leq r'_\alpha \leq a_\alpha$, we can represent these quantities in the form (for details of calculation, see Appendix):

$$F_{l_1 l_2}(r_\alpha, r'_\alpha, \omega) = (-1)^p \frac{2\kappa}{\pi\eta} \left\{ \tilde{h}_{l_1}(x_\alpha) \tilde{j}_{l_2}(x'_\alpha) - \delta_{l_2, l_1 - 2} \left(l_1 - \frac{1}{2} \right) \frac{\pi}{x_\alpha^3} \left(\frac{r'_\alpha}{r_\alpha} \right)^{l_1 - 2} \right\},$$

$$l_2 = l_1 + 2p \geq 0, \quad p = 0, \pm 1, \tag{3.69}$$

$$C_{l_1 l_2}(r_\alpha, r'_\alpha) = \delta_{l_2, l_1 - 1} \frac{1}{r_\alpha^2} \left(\frac{r'_\alpha}{r_\alpha} \right)^{l_1 - 1} + \delta_{r_\alpha, a}\, \delta_{r'_\alpha, a}\, \frac{1}{2a_\alpha^2} \left(\delta_{l_2, l_1 + 1} - \delta_{l_2, l_1 - 1} \right), \ l_2 = l_1 \pm 1 \geq 0, \tag{3.70}$$

where $x'_\alpha = \kappa r'_\alpha$.

For the quantity $P_{1,2}(r_\alpha, a_\alpha, \omega)$, in the domain $r_\alpha \geq a_\alpha$, we obtain

$$P_{1,2}(r_\alpha, a_\alpha, \omega) = \frac{2}{\pi\eta}\, \tilde{h}_1(x_\alpha)\tilde{j}_2(b_\alpha). \tag{3.71}$$

In the particular case $r_\alpha = a_\alpha$ (the observation point lies at the surface of sphere α) and $r'_\alpha = a_\alpha$, relations (3.69)–(3.71) are reduced to the form (see Appendix)

$$F_{l_1 l_2}(a_\alpha, a_\alpha, \omega) = (-1)^p \frac{2\kappa}{\pi\eta}\, \tilde{j}_{l_{max})}(b_\alpha)\tilde{h}_{l_{min}}(b_\alpha), \quad l_2 = l_1 + 2p \geq 0, \quad p = 0, \pm 1, \tag{3.72}$$

$$C_{l_1 l_2}(a_\alpha, a_\alpha) = \frac{1}{2a_\alpha^2}\left(\delta_{l_2, l_1+1} + \delta_{l_2, l_1-1}\right), \qquad l_2 = l_1 \pm 1 \geq 0, \tag{3.73}$$

$$P_{1,2}(a_\alpha, a_\alpha, \omega) = \frac{2}{\pi\eta}\, \tilde{h}_1(b_\alpha)\tilde{j}_2(b_\alpha), \tag{3.74}$$

where $l_{max} = \max(l_1, l_2)$ and $l_{min} = \min(l_1, l_2)$.

For $l_2 = l_1$ and $l_2 = l_1 + 2$, expression (3.72) coincides with the corresponding result obtained in [26]. However, the quantity $\boldsymbol{T}_{l_1 m_1}^{l_2 m_2}(a_\alpha, a_\alpha, \omega)$ defined by relation (3.63) for $r_\alpha = a_\alpha$ and $r'_\alpha = a_\alpha$ is not equal to zero for three values of l_2 [$l_2 = l_1$, $l_1 + 2$, and $l_1 - 2$ (for $l_1 \geq 2$)], while, according to [26], $\boldsymbol{T}_{l_1 m_1}^{l_2 m_2}(a_\alpha, a_\alpha, \omega) \neq 0$ only for $l_2 = l_1$ and $l_2 = l_1 + 2$.

Relations (3.4), (3.6)–(3.8), (3.10), (3.12), (3.24), (3.26), (3.27), (3.29), (3.31), (3.32), (3.58), (3.60) and (3.61) completely determine the function $\boldsymbol{v}(\boldsymbol{r}, \omega)$ in the entire space provided that the induced surface force densities are known. Analogously, relations (2.8), (3.5), (3.9), (3.11), (3.13), (3.15), (3.25), (3.28), (3.30), (3.33), (3.59), and (3.62) uniquely reproduce the original function $p(\boldsymbol{r}, \omega)$ at any point of the space with the exception of the points of the surfaces of the spheres ($r_\alpha = a_\alpha, \quad \alpha = 1, 2, \ldots, N$) where the obtained quantity is equal to the half-sum of the original function given at $\boldsymbol{r} = \boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha + 0$ and $\boldsymbol{r} = \boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha - 0$ because the original function defined as the fluid pressure for $r_\alpha \geq a_\alpha$ and (2.17) for $r_\alpha < a_\alpha$ is discontinuous at the surfaces $r_\alpha = a_\alpha$. With regard for (2.8), (2.17), and (3.5), the fluid pressure at the surfaces of the spheres can be represented as follows:

$$p(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha + 0, \omega) = p^{(0)}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega) + p^{ind}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha + 0, \omega), \tag{3.75}$$

where

$$p^{ind}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha + 0, \omega) = 2p^{ind}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega) - i\omega\rho\,(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha) \cdot \boldsymbol{U}_\alpha(\omega) \tag{3.76}$$

and the quantities $p^{(0)}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega)$ and $p^{ind}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega)$ are defined by the corresponding above-derived expressions for $p^{(0)}(\boldsymbol{r}, \omega)$ and $p^{ind}(\boldsymbol{r}, \omega)$ given at $\boldsymbol{r} = \boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha$.

IV. SYSTEM OF EQUATIONS FOR HARMONICS OF INDUCED SURFACE FORCE DENSITIES

In the previous section, the required distributions of the velocity and pressure fields of the fluid as well as the forces and torques exerted by the fluid on the particles were expressed in terms of harmonics of the induced surface force densities. To determine these harmonics, we use the stick boundary conditions for the fluid velocity at the surfaces of the particles [1,4,5]

$$\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{U}_\alpha(t) + \Big(\boldsymbol{\Omega}_\alpha(t) \times \boldsymbol{r}_\alpha\Big)\Big|_{\boldsymbol{r}=\boldsymbol{R}_\alpha+\boldsymbol{a}_\alpha}, \qquad \alpha = 1, 2, \ldots, N. \tag{4.1}$$

Passing in relations (4.1) to the Fourier transform with respect to the frequency and using representation (3.4) for the fluid velocity, we obtain

$$\boldsymbol{V}_\alpha(\boldsymbol{a}_\alpha, \omega) = \boldsymbol{v}^{ind}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega), \qquad \alpha = 1, 2, \ldots, N, \tag{4.2}$$

where

$$\boldsymbol{V}_\alpha(\boldsymbol{a}_\alpha, \omega) = \boldsymbol{U}_\alpha(\omega) + \Big(\boldsymbol{\Omega}_\alpha(\omega) \times \boldsymbol{a}_\alpha\Big) - \boldsymbol{v}^{(0)}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega) \tag{4.3}$$

is the velocity of the point $\boldsymbol{r}_\alpha = \boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha$ of the surface of sphere α relative to the fluid velocity $\boldsymbol{v}^{(0)}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega)$ at this point in the absence of particles in the fluid.

For each α, we expand the quantities in relations (4.2) in the spherical harmonics $Y_{lm}(\theta, \varphi)$ analogously to expansion (3.19) for the induced surface force densities. Note that representations of the induced velocity and pressure of the fluid in the form (3.24)–(3.28) are, in fact, the expansions of these quantities in the spherical harmonics $Y_{lm}(\theta, \varphi)$ at the surface of the sphere of an arbitrary radius r_α with the center at the point O_α. Therefore, putting $\boldsymbol{r}_\alpha = \boldsymbol{a}_\alpha$ in relations (3.24), (3.26), and (3.27), we immediately obtain the required expansion for the quantity $\boldsymbol{v}^{ind}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega)$ at the surface of particle α in the spherical harmonics $Y_{lm}(\theta, \varphi)$. As a result, we obtain the following system of algebraic equations in the unknown quantities $\boldsymbol{f}_{\beta,lm}(\omega)$:

$$\sum_{\beta=1}^{N} \sum_{l_2 m_2} \boldsymbol{T}^{\beta,l_2 m_2}_{\alpha,l_1 m_1}(\omega) \cdot \boldsymbol{f}_{\beta,l_2 m_2}(\omega) = \boldsymbol{V}_{\alpha,l_1 m_1}(\omega) - \sum_{\beta=1}^{N} \left\{ \boldsymbol{v}^{\beta(r)}_{\alpha,l_1 m_1}(\omega) + \boldsymbol{v}^{\beta(sol)}_{\alpha,l_1 m_1}(\omega) \right\}, \tag{4.4}$$

where

$$\boldsymbol{V}_{\alpha,l_1 m_1}(\omega) = \delta_{l_1,0}\, \delta_{m_1,0} \left\{ \boldsymbol{U}_\alpha(\omega) - \boldsymbol{v}^{inf}(\omega) \right\} + \delta_{l_1,1} \frac{a_\alpha}{\sqrt{\pi}} \left(\boldsymbol{\Omega}_\alpha(\omega) \times \boldsymbol{a}_\alpha \right) - \boldsymbol{v}^{(0)sol}_{\alpha,l_1 m_1}(\omega), \tag{4.5}$$

$$\boldsymbol{v}^{(0)sol}_{\alpha,lm}(\omega) = \frac{1}{2\sqrt{\pi}} \int d\Omega_\alpha\, \boldsymbol{v}^{(0)sol}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha, \omega) Y^*_{lm}(\theta_\alpha, \varphi_\alpha), \tag{4.6}$$

and, to simplify the representation of relations, we introduce the following notation:

$$\begin{aligned}
\boldsymbol{T}^{\beta,l_2 m_2}_{\alpha,l_1 m_1}(\omega) &\equiv \boldsymbol{T}^{l_2 m_2}_{l_1 m_1}(a_\alpha, a_\beta, R_{\alpha\beta}, \omega), &\qquad \boldsymbol{T}^{\alpha,l_2 m_2}_{\alpha,l_1 m_1}(\omega) &\equiv \boldsymbol{T}^{l_2 m_2}_{l_1 m_1}(a_\alpha, a_\alpha, \omega),\\
\boldsymbol{v}^{\beta(r)}_{\alpha,l_1 m_1}(\omega) &\equiv \boldsymbol{v}^{(r)ind}_{\beta,l_1 m_1}(\boldsymbol{R}_\alpha, a_\alpha, \omega), &\qquad \boldsymbol{v}^{\alpha(r)}_{\alpha,l_1 m_1}(\omega) &\equiv \boldsymbol{v}^{(r)ind}_{\alpha,l_1 m_1}(\boldsymbol{R}_\alpha, a_\alpha, \omega),\\
\boldsymbol{v}^{\beta(sol)}_{\alpha,l_1 m_1}(\omega) &\equiv \boldsymbol{v}^{(sol)ind}_{\beta,l_1 m_1}(\boldsymbol{R}_\alpha, a_\alpha, \omega), &\qquad \boldsymbol{v}^{\alpha(sol)}_{\alpha,l_1 m_1}(\omega) &\equiv \boldsymbol{v}^{(sol)ind}_{\alpha,l_1 m_1}(\boldsymbol{R}_\alpha, a_\alpha, \omega),
\end{aligned}$$

where the quantities $\boldsymbol{T}^{l_2 m_2}_{l_1 m_1}(a_\alpha, a_\beta, R_{\alpha\beta}, \omega)$, $\boldsymbol{T}^{l_2 m_2}_{l_1 m_1}(a_\alpha, a_\alpha, \omega)$, $\boldsymbol{v}^{(r)ind}_{\beta,l_1 m_1}(\boldsymbol{R}_\alpha, a_\alpha, \omega)$, $\boldsymbol{v}^{(r)ind}_{\alpha,l_1 m_1}(\boldsymbol{R}_\alpha, a_\alpha, \omega)$, $\boldsymbol{v}^{(sol)ind}_{\beta,l_1 m_1}(\boldsymbol{R}_\alpha, a_\alpha, \omega)$, and $\boldsymbol{v}^{(sol)ind}_{\alpha,l_1 m_1}(\boldsymbol{R}_\alpha, a_\alpha, \omega)$, are defined, respectively, by relations (3.34), (3.63), (3.31), (3.60), (3.32), and (3.61) for $r_\alpha = a_\alpha$.

In the absence of the sum on the right-hand side of Eqs. (4.4) (this sum is absent in the case of the stationary problem and conservative external fields), the tensor $\boldsymbol{T}^{\beta,l_2 m_2}_{\alpha,l_1 m_1}(\omega)$ defines harmonic $(l_1 m_1)$ of the expansion of the fluid velocity at the surface of particle α induced by harmonic $(l_2 m_2)$ of the expansion of the induced surface force density $\boldsymbol{f}_\beta(\boldsymbol{a}_\beta, \omega)$ distributed over the surface of particle β (another particle if $\beta \neq \alpha$ or the same if $\beta = \alpha$). For the stationary case ($\omega = 0$), these quantities were introduced in [30] and called hydrodynamic interaction tensors. Their generalization to the nonstationary case was performed in [25,26]. In what follows, we use this terminology. Equations (4.4) agree with the corresponding equations (3.7) in [26] up to the terms $\boldsymbol{v}^{\beta(r)ind}_{\alpha,l_1 m_1}(\omega)$ and $\boldsymbol{v}^{\beta(sol)ind}_{\alpha,l_1 m_1}(\omega)$, which is caused by extension (2.18) for the stress tensor mentioned above and the zero extension of the external forces to the domains occupied by the particles used in [26].

V. STATIONARY CASE

In the previous sections, we reduced the problem of determination of the velocity and pressure fields of the unbounded viscous fluid induced by an arbitrary number of spheres immersed in it to the solution of the infinite system of linear algebraic equations (4.4) in the harmonics of induced surface force densities $\boldsymbol{f}_{\beta,lm}(\omega)$. In this paper, we consider the important particular case corresponding to the stationary mode, i.e., all quantities are time independent. It is easy to perform the passage to this case using the general relations obtained above, setting $\omega = 0$ in them, and assuming that all quantities are independent of ω [for example, $\boldsymbol{v}^{ind}(\boldsymbol{r}, \omega = 0) \to \boldsymbol{v}^{ind}(\boldsymbol{r})$, $\boldsymbol{T}^{\beta,l_2m_2}_{\alpha,l_1m_1}(\omega = 0) \to \boldsymbol{T}^{\beta,l_2m_2}_{\alpha,l_1m_1}$, etc., i.e., simply omitting the argument ω (or t)]. In view of relations (3.31) and (3.60), we get

$$\begin{array}{llll} \boldsymbol{v}^{(r)ind}_{\beta,l_1m_1}(\boldsymbol{R}_\alpha, \boldsymbol{r}_\alpha) = 0, & \boldsymbol{v}^{\beta(r)ind}_{\alpha,l_1m_1} = 0, & \beta = 1, 2, \ldots, N, \\ \boldsymbol{v}^{(r)ind}_{\beta}(\boldsymbol{r}) = 0, & \boldsymbol{v}^{(V)ind}_{\beta}(\boldsymbol{r}) = \boldsymbol{v}^{(sol)ind}_{\beta}(\boldsymbol{r}), & \beta = 1, 2, \ldots, N. \end{array} \tag{5.1}$$

Thus, in the stationary case, the fluid velocity caused by the induced volume forces is determined only by the solenoidal component of the external force field. This is quite natural because, according to (2.24), in the stationary mode,

$$\boldsymbol{F}^{(V)ind}_{\alpha}(\boldsymbol{r}) = -\Theta(a_\alpha - r_\alpha)\boldsymbol{F}^{(sol)ext}(\boldsymbol{r}). \tag{5.2}$$

Passing in relations (3.52), (3.54), (3.69), and (3.72) to the limit as $\omega \to 0$, we obtain the corresponding relations for the quantities defining the fluid velocity in the stationary mode.

For $\beta \neq \alpha$, $\lim\limits_{\omega \to 0} F_{l_1l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$, where $0 \leq r_\beta \leq a_\beta$ and $l = l_1 + l_2 - 2p$, $p = -1, 0, 1, \ldots, p_{max}$, gives that $F_{l_1l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}) \neq 0$ only if $l = l_1 + l_2$ $(p = 0)$ and $l = l_1 + l_2 + 2$ $(p = -1)$

$$F_{l_1l_2,l_1+l_2}(r_\alpha, r_\beta, R_{\alpha\beta}) = \frac{\sqrt{\pi}}{4\eta R_{\alpha\beta}} \frac{\Gamma\left(l_1 + l_2 + \frac{1}{2}\right)}{\Gamma\left(l_1 + \frac{3}{2}\right)\Gamma\left(l_2 + \frac{3}{2}\right)} \frac{r_\alpha^{l_1} r_\beta^{l_2}}{R_{\alpha\beta}^{l_1+l_2}}, \tag{5.3}$$

$$\begin{aligned} F_{l_1l_2,l_1+l_2+2}(r_\alpha, r_\beta, R_{\alpha\beta}) &= F_{l_1l_2,l_1+l_2}(r_\alpha, r_\beta, R_{\alpha\beta})\left(l_1 + l_2 + \frac{1}{2}\right) \\ &\times \left\{1 - \left(l_1 + l_2 + \frac{3}{2}\right)\left[\frac{1}{l_1 + \frac{3}{2}}\left(\frac{r_\alpha}{R_{\alpha\beta}}\right)^2 + \frac{1}{l_2 + \frac{3}{2}}\left(\frac{r_\beta}{R_{\alpha\beta}}\right)^2\right]\right\} \end{aligned} \tag{5.4}$$

in the domain $r_\alpha \le R_{\alpha\beta} - r_\beta$, and only if $l = l_1 - l_2$ ($l_1 \ge l_2$, $p = l_2$) and $l = l_1 - l_2 - 2$ ($l_1 \ge l_2 + 2, p = l_2 + 1$)

$$F_{l_1 l_2, l_1 - l_2}(r_\alpha, r_\beta, R_{\alpha\beta}) = \frac{\sqrt{\pi}}{4\eta r_\alpha} \frac{\Gamma\left(l_1 + \frac{1}{2}\right)}{\Gamma\left(l_2 + \frac{3}{2}\right)\Gamma\left(l_1 - l_2 + \frac{3}{2}\right)} \frac{r_\beta^{l_2} R_{\alpha\beta}^{l_1 - l_2}}{r_\alpha^{l_1}}, \qquad l_1 \ge l_2, \tag{5.5}$$

$$F_{l_1 l_2, l_1 - l_2 - 2}(r_\alpha, r_\beta, R_{\alpha\beta}) = \frac{\sqrt{\pi}}{4\eta r_\alpha} \frac{\Gamma\left(l_1 - \frac{1}{2}\right)}{\Gamma\left(l_2 + \frac{3}{2}\right)\Gamma\left(l_1 - l_2 - \frac{1}{2}\right)} \frac{r_\beta^{l_2} R_{\alpha\beta}^{l_1 - l_2 - 2}}{r_\alpha^{l_1 - 2}} \left\{1 - \left(l_1 - \frac{1}{2}\right)\right.$$
$$\left. \times \left[\frac{1}{l_1 - l_2 - \frac{1}{2}}\left(\frac{R_{\alpha\beta}}{r_\alpha}\right)^2 + \frac{1}{l_2 + \frac{3}{2}}\left(\frac{r_\beta}{r_\alpha}\right)^2\right]\right\}, \qquad l_1 \ge l_2 + 2 \tag{5.6}$$

in the domain $r_\alpha \ge R_{\alpha\beta} + r_\beta$.

Thus, in the stationary case, the sum over l in relation (3.34) contains only two terms with $l = l_1 + l_2$ and $l = l_1 + l_2 + 2$ for $r_\alpha \le R_{\alpha\beta} - r_\beta$ and with $l = l_1 - l_2 \ge 0$ and $l = l_1 - l_2 - 2 \ge 0$ for $r_\alpha \ge R_{\alpha\beta} + r_\beta$.

For $\beta = \alpha$, for $r_\alpha \ge a_\alpha$, we have

$$F_{l_1 l_2}(r_\alpha, a_\alpha) = \delta_{l_2, l_1}\, F_{l_1 l_1}(r_\alpha, a_\alpha) + \delta_{l_2, l_1 - 2}\, F_{l_1 l_2}(r_\alpha, a_\alpha), \quad l_2 = l_1 + 2n \ge 0,\ n = 0, \pm 1, \tag{5.7}$$

where

$$F_{l_1 l_1}(r_\alpha, a_\alpha) = \frac{1}{(2l_1 + 1)\eta r_\alpha}\left(\frac{a_\alpha}{r_\alpha}\right)^{l_1}, \tag{5.8}$$

$$F_{l_1, l_1 - 2}(r_\alpha, a_\alpha) = \frac{1}{2\eta r_\alpha}\left(\frac{a_\alpha}{r_\alpha}\right)^{l_1 - 2}\left[1 - \left(\frac{a_\alpha}{r_\alpha}\right)^2\right], \qquad l_1 \ge 2. \tag{5.9}$$

According to (5.7)–(5.9),

$$F_{l_1 l_2}(a_\alpha, a_\alpha) = \delta_{l_2, l_1} F_{l_1}(a_\alpha), \qquad l_2 = l_1 + 2n \ge 0, \quad n = 0, \pm 1, \tag{5.10}$$

where

$$F_{l_1}(a_\alpha) \equiv F_{l_1 l_1}(a_\alpha, a_\alpha) = \frac{1}{(2l_1 + 1)\eta a_\alpha}. \tag{5.11}$$

Therefore, the tensor

$$\boldsymbol{T}_{\alpha, l_1 m_1}^{\alpha, l_2 m_2} = \delta_{l_2, l_1}\, \boldsymbol{T}_{\alpha, l_1 m_1}^{\alpha, l_1 m_2} \tag{5.12}$$

is diagonal with respect to the indices l_1 and l_2 [30]. Here,

$$\boldsymbol{T}^{\alpha,l_1m_2}_{\alpha,l_1m_1} = \frac{3\sqrt{\pi}}{(2l_1+1)\xi_\alpha}\boldsymbol{K}^{l_1m_1}_{l_1m_1,00}\,. \tag{5.13}$$

We represent the system of equations (4.4) in the form

$$\sum_{m_2=-l_1}^{l_1}\boldsymbol{T}^{\alpha,l_1m_2}_{\alpha,l_1m_1}\cdot\boldsymbol{f}_{\alpha,l_1m_2} - \boldsymbol{V}_{\alpha,l_1m_1} + \boldsymbol{v}^{\alpha(sol)}_{\alpha,l_1m_1} = -\sum_{\beta\neq\alpha}\left\{\sum_{l_2m_2}\boldsymbol{T}^{\beta,l_2m_2}_{\alpha,l_1m_1}\cdot\boldsymbol{f}_{\beta,l_2m_2} + \boldsymbol{v}^{\beta(sol)}_{\alpha,l_1m_1}\right\} \tag{5.14}$$

separating terms corresponding to the hydrodynamic interaction between different particles on the right-hand side of the equations. In the absence of the external force field, system (5.14) coincides with the corresponding system derived in [30]. Taking into account the obtained explicit form for the quantities $\boldsymbol{T}^{\alpha,l_1m_2}_{\alpha,l_1m_1}$ and $\boldsymbol{T}^{\beta,l_2m_2}_{\alpha,l_1m_1}$, we get for any $l_1, l_2 \geq 0$ [30]

$$\begin{aligned} a_\alpha\boldsymbol{T}^{\alpha,l_2m_2}_{\alpha,l_1m_1} &\sim \delta_{l_2,l_1}\,\sigma^0_{\alpha\beta}, \\ a_\alpha\boldsymbol{T}^{\beta,l_2m_2}_{\alpha,l_1m_1} &\sim \sigma^{l_1+1}_{\alpha\beta}\sigma^{l_2}_{\beta\alpha}\,. \end{aligned} \tag{5.15}$$

This enables us to seek a solution of system (5.14) by the method of successive approximations. As the zeroth iteration, we use system (5.14) with the zeroth right-hand side, which corresponds to the absence of interaction between the particles (the induced surface force density on the surface of particle α is determined by the characteristics of the fluid and this particle and independent of the characteristics of the rest particles). The solution of system (5.14) in the form of a series, each term of which, in fact, determines the contribution of a separate iteration to the total solution is given in [30]. Moreover, the final result is expressed in terms of the tensor $\boldsymbol{T}^{\beta,l_2m_2}_{\alpha,l_1m_1}$ and the inverse tensor $\tilde{\boldsymbol{T}}^{\alpha,l_1m_2}_{\alpha,l_1m_1}$ defined by the condition

$$\sum_{m_3=-l_1}^{l_1}\tilde{\boldsymbol{T}}^{\alpha,l_1m_3}_{\alpha,l_1m_1}\cdot\boldsymbol{T}^{\alpha,l_1m_2}_{\alpha,l_1m_3} = \delta_{m_1,m_2}\boldsymbol{I}. \tag{5.16}$$

Taking relation (5.13) into account, the inverse tensor $\tilde{\boldsymbol{T}}^{\alpha,l_1m_2}_{\alpha,l_1m_1}$ can be represented in the form

$$\tilde{\boldsymbol{T}}^{\alpha,l_1m_2}_{\alpha,l_1m_1} = \frac{(2l_1+1)\xi_\alpha}{3\sqrt{\pi}}\tilde{\boldsymbol{K}}^{l_1m_2}_{l_1m_1,00} \tag{5.17}$$

where $\tilde{\boldsymbol{K}}^{l_2m_2}_{l_1m_1,00}$ is the tensor inverse to the tensor $\boldsymbol{K}^{l_2m_2}_{l_1m_1,00}$ defined by the condition

$$\sum_{m_3=-l_1}^{l_1}\tilde{\boldsymbol{K}}^{l_1m_3}_{l_1m_1,00}\cdot\boldsymbol{K}^{l_1m_2}_{l_1m_3,00} = \delta_{m_1,m_2}\boldsymbol{I}. \tag{5.18}$$

However, the explicit form for the tensor $\tilde{\boldsymbol{T}}^{\alpha,l_1m_2}_{\alpha,l_1m_1}$ is not given in [30] except for $l_1 = 0$. Furthermore, it is stated that this tensor always exists due to the uniqueness of the solution of the Stokes equation. Being the central point for the determination of the solution in [30], the derivation of the forces and torques exerted by the fluid on particles is based on this statement. As a result, these forces and torques are also expressed in terms of the inverse tensor $\tilde{\boldsymbol{T}}^{\alpha,l_1m_2}_{\alpha,l_1m_1}$. For this reason, in the present paper, we dwell on the derivation of a similar solution in detail. To this end, we represent $\boldsymbol{f}_{\alpha,l_1m_1}$ in the form

$$\boldsymbol{f}_{\alpha,l_1m_1} = \sum_{n=0}^{\infty} \boldsymbol{f}^{(n)}_{\alpha,l_1m_1}, \tag{5.19}$$

where $\boldsymbol{f}^{(n)}_{\alpha,l_1m_1}$ is a solution of system (5.14) corresponding the nth iteration.

In a similar form, we represent the force $\boldsymbol{F}_\alpha$ and the torque $\boldsymbol{T}_\alpha$ exerted by the fluid on particle α (as well as the induced velocity and pressure of the fluid)

$$\boldsymbol{F}_\alpha = \sum_{n=0}^{\infty} \boldsymbol{F}^{(n)}_\alpha, \tag{5.20}$$

$$\boldsymbol{T}_\alpha = \sum_{n=0}^{\infty} \boldsymbol{T}^{(n)}_\alpha, \tag{5.21}$$

where $\boldsymbol{F}^{(n)}_\alpha$ and $\boldsymbol{T}^{(n)}_\alpha$ are the force and the torque corresponding the nth iteration

$$\boldsymbol{F}^{(n)}_\alpha = -\boldsymbol{f}^{(n)}_{\alpha,00}\,, \tag{5.22}$$

$$\boldsymbol{T}^{(n)}_\alpha = -\frac{a_\alpha}{\sqrt{3}} \sum_{m=-1}^{1} \left(\boldsymbol{e}_m \times \boldsymbol{f}^{(n)}_{\alpha,1m}\right). \tag{5.23}$$

A. $n = 0$. Noninteracting Particles

In this approximation (zero iteration), the infinite system of equations (5.14) is reduced to the collection of independent systems of equations for each particle α and each l_1

$$\sum_{m_2=-l_1}^{l_1} \boldsymbol{T}^{\alpha,l_1m_2}_{\alpha,l_1m_1} \cdot \boldsymbol{f}^{(0)}_{\alpha,l_1m_2} - \boldsymbol{V}_{\alpha,l_1m_1} + \boldsymbol{v}^{\alpha(sol)}_{\alpha,l_1m_1} = 0. \tag{5.24}$$

For $l_1 = 0$,

$$\boldsymbol{K}^{00}_{00,00} = \frac{1}{3\sqrt{\pi}}, \qquad \boldsymbol{T}^{00}_{\alpha,00} = \xi_\alpha^{-1}\boldsymbol{I}, \tag{5.25}$$

and, according to (5.18) and (5.16), we have

$$\tilde{\boldsymbol{K}}^{00}_{00,00} = 3\sqrt{\pi}, \qquad \tilde{\boldsymbol{T}}^{\alpha,00}_{\alpha,00} = \xi_\alpha \boldsymbol{I}. \tag{5.26}$$

We represent the solution $\boldsymbol{f}^{(0)}_{\alpha,00}$ of system (5.24) with $l_1 = 0$ in the form

$$\boldsymbol{f}^{(0)}_{\alpha,00} = \boldsymbol{f}^{(t,0)}_{\alpha,00} + \boldsymbol{f}^{(ext,0)}_{\alpha,00}, \tag{5.27}$$

where the harmonics $\boldsymbol{f}^{(t,0)}_{\alpha,00}$ and $\boldsymbol{f}^{(ext,0)}_{\alpha,00}$ associated, respectively, with the relative translational motion of particle α with the velocity

$$\boldsymbol{U}^{(t)}_\alpha = \boldsymbol{U}_\alpha - \boldsymbol{v}^{inf} \tag{5.28}$$

and the external force field are determined as follows:

$$\boldsymbol{f}^{(t,0)}_{\alpha,00} = \xi_\alpha \boldsymbol{U}^{(t)}_\alpha, \tag{5.29}$$

$$\boldsymbol{f}^{(ext,0)}_{\alpha,00} = -\xi_\alpha \overline{\boldsymbol{v}^{(0)sol}(\boldsymbol{r})}^{S_\alpha} + \tilde{\boldsymbol{F}}^{(sol)ext}_\alpha. \tag{5.30}$$

Here, $\overline{\boldsymbol{v}^{(0)sol}(\boldsymbol{r})}^{S_\alpha} = \boldsymbol{v}^{(0)sol}_{\alpha,00}$ is the fluid velocity induced by the solenoidal component of the external force field in the absence of particles averaged over the surface of particle α and

$$\tilde{\boldsymbol{F}}^{(sol)ext}_\alpha = \int\limits_{V_\alpha} d\boldsymbol{r}\, \boldsymbol{F}^{(sol)ext}(\boldsymbol{r}) \tag{5.31}$$

is the force acting by the solenoidal component $\boldsymbol{F}^{(sol)ext}(\boldsymbol{r})$ of the external force field $\boldsymbol{F}^{ext}(\boldsymbol{r})$ on the fluid occupying the volume V_α.

Analogously to (5.27), in what follows, for any nth iteration, we represent the quantity $\boldsymbol{f}^{(n)}_{\alpha,lm}$ as the superposition of the components

$$\boldsymbol{f}^{(n)}_{\alpha,lm} = \boldsymbol{f}^{(t,n)}_{\alpha,lm} + \boldsymbol{f}^{(r,n)}_{\alpha,lm} + \boldsymbol{f}^{(ext,n)}_{\alpha,lm}, \qquad n = 0, 1, 2, \ldots \tag{5.32}$$

associated, respectively, with the translational motion of particle α, its rotation, and the external force field (as is shown in what follows, in the particular case $l = 1$, $\boldsymbol{f}^{(n)}_{\alpha,1m}$ may be represented as a superposition of four components). To separate the contributions caused by the translational motion of particles, their rotation, and the external force field to the induced velocity and pressure of the fluid as well as to the forces and torques exerted by the fluid on particles, we also use analogous representations for these quantities. According to (5.27) and (5.32), for zero iteration, we have

$$\boldsymbol{f}^{(r,0)}_{\alpha,00} = 0\,. \tag{5.33}$$

Now we consider the case $l_1 = 1$. According to (5.24), we have

$$\sum_{m_2=-1}^{1} \boldsymbol{K}^{1m_2}_{1m_1,00} \cdot \boldsymbol{f}^{(0)}_{\alpha,1m_2} = \frac{\xi_\alpha}{\sqrt{\pi}} \left\{ \frac{a_\alpha}{\sqrt{3}} \left(\boldsymbol{\Omega}_\alpha \times \boldsymbol{e}^*_{m_1} \right) - \boldsymbol{u}^{sol}_{\alpha,1m_1} \right\}, \tag{5.34}$$

where

$$\boldsymbol{u}^{sol}_{\alpha,1m} = \boldsymbol{v}^{(0)sol}_{\alpha,1m} + \boldsymbol{v}^{\alpha(sol)}_{\alpha,1m}\,. \tag{5.35}$$

Using relation (3.48) for $\boldsymbol{K}^{l_2m_2}_{l_1m_1,00}$ and setting $l_2 = l_1 = 1$ in it, we obtain

$$\det \boldsymbol{K}^{1m_2}_{1m_1,00} = 0\,. \tag{5.36}$$

Therefore, the tensors $\boldsymbol{K}^{1m_2}_{1m_1,00}$ and $\boldsymbol{T}^{\alpha,1m_2}_{\alpha,1m_1}$ are degenerate and, hence, the inverse tensors $\tilde{\boldsymbol{K}}^{1m_2}_{1m_1,00}$ and $\tilde{\boldsymbol{T}}^{\alpha,1m_2}_{\alpha,1m_1}$ do not exist or they should be defined in another way [23] than by relations (5.18) and (5.16). Thus, even for the case of noninteracting particles, it is difficult to interpret the relation for $\boldsymbol{f}_{\alpha,1m}$ obtained in [30] and expressed in terms of the nonexistent (in the ordinary sense) inverse tensor $\tilde{\boldsymbol{T}}^{\alpha,1m_2}_{\alpha,1m_1}$. Furthermore, with regard for the interaction between particles, not only harmonics with $l_1 = 1$ but all harmonics $\boldsymbol{f}_{\alpha,l_1m_1}$ are expressed in terms of $\tilde{\boldsymbol{T}}^{\alpha,1m_2}_{\alpha,1m_1}$ [relation (38) in [30]]. To understand the reason leading to the impossibility of the existence of $\tilde{\boldsymbol{T}}^{\alpha,1m_2}_{\alpha,1m_1}$, we investigate the system of equations (5.34) in detail. To this end, first, we decompose all vectors in (5.34) into the independent unit vectors $\boldsymbol{e}_0$ and $\boldsymbol{e}_{\pm1}$ defined by relations (3.23) taking into account that any vector $\boldsymbol{a} \equiv (a_x, a_y, a_z)$ may be represented in terms of these unit vectors in the form $\boldsymbol{a} \equiv (a_{+1}, a_{-1}, a_0)$, i.e.,

$$\boldsymbol{a} = a_0\boldsymbol{e}_0 - (a_{-1}\boldsymbol{e}_1 + a_{+1}\boldsymbol{e}_{-1})\,, \tag{5.37}$$

where $a_{\pm 1} = (ia_y \pm a_x)/\sqrt{2}$.

Going from $\boldsymbol{\Omega}_\alpha \equiv (\Omega_{\alpha x}, \Omega_{\alpha y}, \Omega_{\alpha z})$, $\boldsymbol{f}^{(0)}_{\alpha,1m} \equiv (f^{(0)}_{\alpha,1m,x}, f^{(0)}_{\alpha,1m,y}, f^{(0)}_{\alpha,1m,z})$, and $\boldsymbol{u}^{sol}_{\alpha,1m} \equiv (u^{sol}_{\alpha,1m,x}, u^{sol}_{\alpha,1m,y}, u^{sol}_{\alpha,1m,z})$ to $\boldsymbol{\Omega}_\alpha \equiv (\Omega_{\alpha,+1}, \Omega_{\alpha,-1}, \Omega_{\alpha,0})$, $\boldsymbol{f}^{(0)}_{\alpha,m} \equiv (f^{(0)}_{\alpha,m+1}, f^{(0)}_{\alpha,m-1}, f^{(0)}_{\alpha,m0})$, and $\boldsymbol{u}^{sol}_{\alpha,m} \equiv (u^{sol}_{\alpha,m+1}, u^{sol}_{\alpha,m-1}, u^{sol}_{\alpha,m0})$, we reduce the system of nine equations (5.34) to three independent systems

$$\left\{\begin{aligned} 2f^{(0)}_{\alpha,00} - f^{(0)}_{\alpha,+1+1} - f^{(0)}_{\alpha,-1-1} &= 10\xi_\alpha u^{sol}_{\alpha,00} \\ f^{(0)}_{\alpha,00} - 3f^{(0)}_{\alpha,+1+1} + 2f^{(0)}_{\alpha,-1-1} &= -i\varepsilon_\alpha \Omega_{\alpha,0} + 10\xi_\alpha u^{sol}_{\alpha,+1+1} \\ f^{(0)}_{\alpha,00} + 2f^{(0)}_{\alpha,+1+1} - 3f^{(0)}_{\alpha,-1-1} &= i\varepsilon_\alpha \Omega_{\alpha,0} + 10\xi_\alpha u^{sol}_{\alpha,-1-1}, \end{aligned}\right. \tag{5.38}$$

$$\left\{\begin{aligned} 4f^{(0)}_{\alpha,0-1} + f^{(0)}_{\alpha,+10} &= i\varepsilon_\alpha \Omega_{\alpha,-1} - 10\xi_\alpha u^{sol}_{\alpha,0-1} \\ f^{(0)}_{\alpha,0-1} + 4f^{(0)}_{\alpha,+10} &= i\varepsilon_\alpha \Omega_{\alpha,-1} - 10\xi_\alpha u^{sol}_{\alpha,+10}\,, \end{aligned}\right. \tag{5.39}$$

$$\left\{\begin{aligned} 4f^{(0)}_{\alpha,0+1} + f^{(0)}_{\alpha,-10} &= -i\varepsilon_\alpha \Omega_{\alpha,+1} - 10\xi_\alpha u^{sol}_{\alpha,0+1} \\ f^{(0)}_{\alpha,0+1} + 4f^{(0)}_{\alpha,-10} &= -i\varepsilon_\alpha \Omega_{\alpha,+1} - 10\xi_\alpha u^{sol}_{\alpha,-10}\,, \end{aligned}\right. \tag{5.40}$$

where $\varepsilon_\alpha = 10a_\alpha\xi_\alpha/\sqrt{3}$, and

$$\begin{aligned} f^{(0)}_{\alpha,+1-1} &= -\frac{10}{3}\,\xi_\alpha u^{sol}_{\alpha,+1-1}, \\ f^{(0)}_{\alpha,-1+1} &= -\frac{10}{3}\,\xi_\alpha u^{sol}_{\alpha,-1+1}. \end{aligned} \tag{5.41}$$

The solutions of systems (5.39) and (5.40) have the form

$$\begin{aligned} f^{(0)}_{\alpha,0+1} &= -i\frac{\varepsilon_\alpha}{5}\Omega_{\alpha,+1} + \frac{2}{3}\,\xi_\alpha \left(u^{sol}_{\alpha,-10} - 4u^{sol}_{\alpha,0+1}\right), \\ f^{(0)}_{\alpha,0-1} &= i\frac{\varepsilon_\alpha}{5}\Omega_{\alpha,-1} + \frac{2}{3}\,\xi_\alpha \left(u^{sol}_{\alpha,+10} - 4u^{sol}_{\alpha,0-1}\right), \\ f^{(0)}_{\alpha,+10} &= i\frac{\varepsilon_\alpha}{5}\Omega_{\alpha,-1} + \frac{2}{3}\,\xi_\alpha \left(u^{sol}_{\alpha,0-1} - 4u^{sol}_{\alpha,+10}\right), \\ f^{(0)}_{\alpha,-10} &= -i\frac{\varepsilon_\alpha}{5}\Omega_{\alpha,+1} + \frac{2}{3}\,\xi_\alpha \left(u^{sol}_{\alpha,0+1} - 4u^{sol}_{\alpha,-10}\right). \end{aligned} \tag{5.42}$$

The determinant of system (5.38) is equal to zero. Indeed, it is easy to see that the left-hand side of the first equation of system (5.38) is equal to the sum of the left-hand sides of the second and third equations of this system. For the consistency of system (5.38), the right-hand sides of the equations of this system must satisfy the following condition:

$$\sum_{m=-1}^{1} \boldsymbol{e}_m \cdot \boldsymbol{u}^{sol}_{\alpha,1m} = 0\,. \tag{5.43}$$

Using relations (3.61) and (4.6), we verify the validity of this condition because

$$\sum_{m=-1}^{1} \boldsymbol{e}_m \cdot \boldsymbol{v}^{(0)sol}_{\alpha,1m} = 0, \tag{5.44}$$

$$\sum_{m=-1}^{1} \boldsymbol{e}_m \cdot \boldsymbol{v}^{\alpha(sol)}_{\alpha,1m} = 0\,. \tag{5.45}$$

Furthermore, relations (5.44) and (5.45) are also true in the general nonstationary case where $\omega \neq 0$ including any values of r_α and not only for $r_\alpha = a_\alpha$.

Therefore, system (5.38) and, hence the original system (5.34), has an infinite number of solutions instead of a single solution as it is stated in [30].

This result is quite natural. Indeed, the problem of determination of $\boldsymbol{f}_{\alpha,lm}$ with the use of boundary conditions (4.1) means that we try to represent the unknown induced surface force densities $\boldsymbol{f}_\alpha(\boldsymbol{a}_\alpha)$ in terms of the known fluid velocity at the surfaces of the particles. According to the continuity equation (2.2) for the incompressible fluid, the fluid velocity is a solenoidal vector. Thus, within the framework of this approach, the required quantity $\boldsymbol{f}_\alpha(\boldsymbol{a}_\alpha)$ can be determined only up to an arbitrary potential vector. The potential component of the induced surface force must make no contribution to the fluid velocity. Therefore, instead of the first equation in system (5.38), as an additional equation, we can use the condition of the absence of the potential component of the induced surface force density $\boldsymbol{f}_\alpha(\boldsymbol{a}_\alpha)$.

In the general case where the interaction between the particles is taken into account, the conditions of the absence of the potential components of the induced surface forces can be written in the form

$$Y_\alpha = 0, \qquad \alpha = 1, 2, \ldots, N, \tag{5.46}$$

where

$$Y_\alpha = \sum_{m=-1}^{1} \boldsymbol{e}_m \cdot \boldsymbol{f}_{\alpha,1m} = f_{\alpha,00} + f_{\alpha,+1+1} + f_{\alpha,-1-1}, \quad \alpha = 1, 2, \ldots, N. \tag{5.47}$$

In the approximation of noninteracting particles considered in this section, these conditions have the form

$$Y_\alpha^{(0)} \equiv \sum_{m=-1}^{1} \boldsymbol{e}_m \cdot \boldsymbol{f}_{\alpha,1m}^{(0)} = f_{\alpha,00}^{(0)} + f_{\alpha,+1+1}^{(0)} + f_{\alpha,-1-1}^{(0)} = 0, \quad \alpha = 1,2,\ldots,N, \tag{5.48}$$

where the superscript 0 in $Y_\alpha^{(0)}$, just as in $\boldsymbol{f}_{\alpha,lm}^{(0)}$ and other quantities, stands for zero approximation.

In the case where both the solenoidal and potential components of the induced surface forces should be determined, a certain additional equation linearly independent of the second and third equations of system (5.38) should be formulated.

To derive additional equations with regard for the interaction between the particles in the fluid, we consider relation (3.5) at $\boldsymbol{r} = \boldsymbol{R}_\alpha$ and equate it to relation (2.17) given at $\boldsymbol{r} = \boldsymbol{R}_\alpha$. As result, we get the relations

$$Y_\alpha = 4\pi\sqrt{3}\, a_\alpha^2 \sum_{\beta\neq\alpha} \left\{ p_\beta^{(V)ind}(\boldsymbol{R}_\alpha) + p_\beta^{(S)ind}(\boldsymbol{R}_\alpha) \right\}, \quad \alpha = 1,2,\ldots,N. \tag{5.49}$$

In the derivation of Eqs. (5.49), we used the explicit form for the quantities $\boldsymbol{D}_{l_1 m_1}^{l_2 m_2}(0, r'_\alpha)$, where $r'_\alpha \leq a_\alpha$, and obtained that $p_\alpha^{(V)ind}(\boldsymbol{R}_\alpha) = 0$.

Putting $r_\alpha = \varepsilon$, where $\varepsilon \to +0$, in relations (3.25), (3.28), (3.38), and (3.40), we can represent the quantity $p_\beta^{(S)ind}(\boldsymbol{R}_\alpha)$ in the form (for details of calculation of the required quantities for $r_\alpha = 0$, see Appendix)

$$p_\beta^{(S)ind}(\boldsymbol{R}_\alpha) = \frac{1}{4\pi R_{\alpha\beta}^2} \Big\{ \Big(\boldsymbol{n}_{\alpha\beta} \cdot \boldsymbol{f}_{\beta,00} \Big) + 4\pi \sum_{l_2=1}^{\infty} \sum_{m_2=-l_2}^{l_2} \sigma_{\beta\alpha}^{l_2} \sum_{m=-(l_2+1)}^{l_2+1} \boldsymbol{W}_{00,l_2+1,m}^{l_2 m_2} \cdot \boldsymbol{f}_{\beta,l_2 m_2} \times Y_{l_2+1,m}(\Theta_{\alpha\beta}, \Phi_{\alpha\beta}) \Big\}, \tag{5.50}$$

where $\boldsymbol{n}_{\alpha\beta} = \boldsymbol{R}_{\alpha\beta}/R_{\alpha\beta}$.

Thus, in the general case, Eqs. (5.49) differ from Eqs. (5.46) corresponding to the absence of the potential components of the induced surface forces by the nonzero right-hand side. However, in the approximation of noninteracting particles, equating the right-hand of Eqs. (5.49) to zero, we obtain Eqs. (5.48), which means that the induced surface forces have no potential components in this approximation.

Using Eq. (5.49) and the second and third equations of system (5.38), we obtain the system of three equations with nonzero determinant, the solution of which has the form

$$
\begin{aligned}
f^{(0)}_{\alpha,-1-1} &= -i\frac{\varepsilon_\alpha}{5}\Omega_{\alpha,0} - \frac{2}{3}\,\xi_\alpha\left(4u^{sol}_{\alpha,-1-1} + u^{sol}_{\alpha,+1+1}\right), \\
f^{(0)}_{\alpha,+1+1} &= i\frac{\varepsilon_\alpha}{5}\Omega_{\alpha,0} - \frac{2}{3}\,\xi_\alpha\left(4u^{sol}_{\alpha,+1+1} + u^{sol}_{\alpha,-1-1}\right), \\
f^{(0)}_{\alpha,00} &= \frac{10}{3}\,\xi_\alpha\left(u^{sol}_{\alpha,+1+1} + u^{sol}_{\alpha,-1-1}\right).
\end{aligned}
\tag{5.51}
$$

Returning in relations (5.41), (5.42), and (5.51) to the quantities $\boldsymbol{f}^{(0)}_{\alpha,1m}$, we get

$$
\boldsymbol{f}^{(0)}_{\alpha,1m} = \boldsymbol{f}^{(r,0)}_{\alpha,1m} + \boldsymbol{f}^{(ext,0)}_{\alpha,1m}, \tag{5.52}
$$

where

$$
\boldsymbol{f}^{(r,0)}_{\alpha,1m} = \frac{\sqrt{3}}{2}\frac{\xi^R_\alpha}{a_\alpha}\left(\boldsymbol{\Omega}_\alpha \times \boldsymbol{e}^*_m\right), \tag{5.53}
$$

$$
\boldsymbol{f}^{(ext,0)}_{\alpha,1m} = -\frac{2}{3}\,\xi_\alpha\left(4\boldsymbol{u}^{sol}_{\alpha,1m} + \left(\boldsymbol{u}^{sol}_{\alpha,1m}\right)^T\right), \tag{5.54}
$$

and $\xi^R_\alpha = 8\pi\eta a^3_\alpha$ is the Stokes friction coefficient for a rotating sphere of radius a_α. The quantities $\left(\boldsymbol{u}^{sol}_{\alpha,1m}\right)^T$ are defined as follows: We consider three vectors $\boldsymbol{b}_m \equiv (b_{m,x}, b_{m,y}, b_{m,z})$, where $m = 0, \pm 1$. According to (5.37), we go to the vectors $\boldsymbol{b}_m \equiv (b_{m+1}, b_{m-1}, b_{m0})$ and represent $\boldsymbol{b}_m$ in the form

$$
\boldsymbol{b}_{m_1} = \sum_{m_2=-1}^{1} \boldsymbol{B}_{m_1 m_2}\cdot \boldsymbol{e}_{m_2}, \qquad m_1 = 0, \pm 1, \tag{5.55}
$$

where

$$
\boldsymbol{B}_{m_1 m_2} = \begin{pmatrix} b_{00} & -b_{0-1} & -b_{0+1} \\ b_{+10} & -b_{+1-1} & -b_{+1+1} \\ b_{-10} & -b_{-1-1} & -b_{-1+1} \end{pmatrix}, \qquad m_1, m_2 = 0, \pm 1. \tag{5.56}
$$

Then $\boldsymbol{b}^T_m$ is the vector defined as follows

$$
\boldsymbol{b}^T_{m_1} = \sum_{m_1=-1}^{1} \boldsymbol{B}^T_{m_1 m_2}\cdot \boldsymbol{e}_{m_2}, \qquad m_1 = 0, \pm 1, \tag{5.57}
$$

where $\boldsymbol{B}^T_{m_1 m_2}$ is the matrix transposed to the matrix $\boldsymbol{B}_{m_1 m_2}$.

By virtue of (5.52), the translational motion of particles makes no contribution to the harmonic $\boldsymbol{f}^{(0)}_{\alpha,1m}$ $\quad [\,\boldsymbol{f}^{(t,0)}_{\alpha,1m} = 0\,]$.

In view of relation (5.54) for $\boldsymbol{f}^{(ext,0)}_{\alpha,1m}$, a similar representation may be given for any right-hand side of system (5.34) for which this system is consistent. Indeed, we can analogously represent the quantity $\boldsymbol{f}^{(r,0)}_{\alpha,1m}$

$$\boldsymbol{f}^{(r,0)}_{\alpha,1m} = \frac{2a_\alpha \xi_\alpha}{3\sqrt{3}} \left\{ 4\boldsymbol{b}^{(r)}_{\alpha,m} + \left(\boldsymbol{b}^{(r)}_{\alpha,m}\right)^T \right\}, \tag{5.58}$$

where the vector $\boldsymbol{b}^{(r)}_{\alpha,m} = (\boldsymbol{\Omega}_\alpha \times \boldsymbol{e}^*_m)$ can be rewritten in the form (5.55) with the asymmetric tensor $\boldsymbol{B}^{(r)}_{\alpha,m_1m_2}$

$$\boldsymbol{B}^{(r)}_{\alpha,m_1m_2} = \begin{pmatrix} 0 & -i\Omega_{\alpha,-1} & i\Omega_{\alpha,+1} \\ i\Omega_{\alpha,-1} & 0 & -i\Omega_{\alpha,0} \\ -i\Omega_{\alpha,+1} & i\Omega_{\alpha,0} & 0 \end{pmatrix}, \qquad m_1, m_2 = 0, \pm 1. \tag{5.59}$$

According to (5.59), we have $\left(\boldsymbol{B}^{(r)}_{\alpha,m_1m_2}\right)^T = -\boldsymbol{B}^{(r)}_{\alpha,m_1m_2}$, which gives $\left(\boldsymbol{b}^{(r)}_{\alpha,m}\right)^T = -\boldsymbol{b}^{(r)}_{\alpha,m}$.

For $l_1 \geq 2$, according to (5.24), we have

$$\sum_{m_2=-l_1}^{l_1} \boldsymbol{T}^{\alpha,l_1m_2}_{\alpha,l_1m_1} \cdot \boldsymbol{f}^{(0)}_{\alpha,l_1m_2} = -\boldsymbol{u}^{sol}_{\alpha,l_1m_1}, \tag{5.60}$$

where

$$\boldsymbol{u}^{sol}_{\alpha,lm} = \boldsymbol{v}^{(0)sol}_{\alpha,lm} + \boldsymbol{v}^{\alpha(sol)}_{\alpha,lm}. \tag{5.61}$$

Assuming the existence of the inverse matrix $\tilde{\boldsymbol{K}}^{l_1m_2}_{l_1m_1,00}$ defined by relation (5.18) for $l_1 \geq 2$, we obtain

$$\boldsymbol{f}^{(0)}_{\alpha,l_1m_1} \equiv \boldsymbol{f}^{(ext,0)}_{\alpha,l_1m_1} = -\sum_{m_2=-l_1}^{l_1} \tilde{\boldsymbol{T}}^{\alpha,l_1m_2}_{\alpha,l_1m_1} \cdot \boldsymbol{u}^{sol}_{\alpha,l_1m_2}, \tag{5.62}$$

i.e., in the approximation of noninteracting particles, the harmonics of the induced surface force densities $\boldsymbol{f}^{(0)}_{\alpha,l_1m_1}$ for $l_1 \geq 2$ are not equal to zero only if the external force field has the solenoidal component.

Substituting the obtained solutions (5.27), (5.29), (5.30), (5.52)–(5.54), and (5.62) into (3.58), (3.59), (3.29), and (3.30) and using (3.34), (3.35), and (3.43)–(3.51), we obtain the following relations for the t, r, and ext components of the harmonics of the fluid velocity and pressure caused by translational motion of particles, their rotation, and the external (solenoidal) force field, respectively, in the approximation of noninteracting particles:

$$\boldsymbol{v}^{(S,t,0)ind}_{\alpha,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \frac{a_\alpha}{r_\alpha}\left\{\delta_{l_1,0}\,\delta_{m_1,0}\,\boldsymbol{I} + \delta_{l_1,2}\,\frac{1}{4}\sqrt{\frac{3}{10}}\left[1-\left(\frac{a_\alpha}{r_\alpha}\right)^2\right]\boldsymbol{K}_{m_1}\right\}\cdot\boldsymbol{U}^{(t)}_\alpha, \tag{5.63}$$

$$\boldsymbol{v}^{(S,r,0)ind}_{\alpha,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \delta_{l_1,1}\,\frac{1}{\sqrt{3}}\frac{a_\alpha^3}{r_\alpha^2}\left(\boldsymbol{\Omega}_\alpha\times\boldsymbol{e}^*_{m_1}\right), \tag{5.64}$$

$$\boldsymbol{v}^{(S,ext,0)ind}_{\alpha,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \sum_{m_2=-l_1}^{l_1}\boldsymbol{T}^{l_1m_2}_{l_1m_1}(r_\alpha, a_\alpha)\cdot\boldsymbol{f}^{(ext,0)}_{\alpha,l_1m_2}, \tag{5.65}$$

$$p^{(S,t,0)ind}_{\alpha,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \left(1-\frac{1}{2}\,\delta_{r_\alpha,a_\alpha}\right)\,\delta_{l_1,1}\,\frac{\sqrt{3}}{2}\eta\frac{a_\alpha}{r_\alpha^2}\,\boldsymbol{e}^*_{m_1}\cdot\boldsymbol{U}^{(t)}_\alpha, \tag{5.66}$$

$$p^{(S,ext,0)ind}_{\alpha,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \sum_{m_2=-l_1}^{l_1}\boldsymbol{D}^{l_1m_2}_{l_1m_1}(r_\alpha, a_\alpha)\cdot\boldsymbol{f}^{(ext,0)}_{\alpha,l_1m_2}, \tag{5.67}$$

$$\boldsymbol{v}^{(S,t,0)ind}_{\beta,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \xi_\beta\boldsymbol{T}^{00}_{l_1m_1}(r_\alpha, a_\beta, \boldsymbol{R}_{\alpha\beta})\cdot\boldsymbol{U}^{(t)}_\beta, \qquad \beta\neq\alpha, \tag{5.68}$$

$$\boldsymbol{v}^{(S,r,0)ind}_{\beta,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \frac{\sqrt{3}}{2}\frac{\xi^R_\beta}{a_\beta}\sum_{m_2=-1}^{1}\boldsymbol{T}^{1m_2}_{l_1m_1}(r_\alpha, a_\beta, \boldsymbol{R}_{\alpha\beta})\cdot\left(\boldsymbol{\Omega}_\beta\times\boldsymbol{e}^*_{m_2}\right), \qquad \beta\neq\alpha, \tag{5.69}$$

$$\boldsymbol{v}^{(S,ext,0)ind}_{\beta,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \sum_{l_2m_2}\boldsymbol{T}^{l_2m_2}_{l_1m_1}(r_\alpha, a_\beta, \boldsymbol{R}_{\alpha\beta})\cdot\boldsymbol{f}^{(ext,0)}_{\beta,l_2m_2}, \qquad \beta\neq\alpha, \tag{5.70}$$

$$p^{(S,t,0)ind}_{\beta,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \xi_\beta\boldsymbol{D}^{00}_{l_1m_1}(r_\alpha, a_\beta, \boldsymbol{R}_{\alpha\beta})\cdot\boldsymbol{U}^{(t)}_\beta, \qquad \beta\neq\alpha, \tag{5.71}$$

$$p^{(S,ext,0)ind}_{\beta,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = \sum_{l_2m_2}\boldsymbol{D}^{l_2m_2}_{l_1m_1}(r_\alpha, a_\beta, \boldsymbol{R}_{\alpha\beta})\cdot\boldsymbol{f}^{(ext,0)}_{\beta,l_2m_2}, \qquad \beta\neq\alpha, \tag{5.72}$$

$$p^{(S,r,0)ind}_{\beta,l_1m_1}(\boldsymbol{R}_\alpha, r_\alpha) = 0, \qquad \beta = 1,2,\ldots,N. \tag{5.73}$$

According to (5.73), the rotation of the particles has no effect on the fluid pressure

$$p^{(S,r,0)ind}_\beta(\boldsymbol{r}) = 0, \qquad \beta = 1,2,\ldots,N. \tag{5.74}$$

Substituting (5.63), (5.64), and (5.66) into (3.24) and (3.25) and taking relation (3.76) into account, we obtain the following space distributions for the corresponding components of the velocity and pressure fields of the fluid:

$$\boldsymbol{v}^{(S,t,0)ind}_\alpha(\boldsymbol{R}_\alpha+\boldsymbol{r}_\alpha) = \frac{a_\alpha}{r_\alpha}\left\{1-\frac{1}{4}\left[1-\left(\frac{a_\alpha}{r_\alpha}\right)^2\right](\boldsymbol{I}-3\boldsymbol{n}_\alpha\boldsymbol{n}_\alpha)\right\}\cdot\boldsymbol{U}^{(t)}_\alpha, \tag{5.75}$$

$$\boldsymbol{v}^{(S,r,0)ind}_\alpha(\boldsymbol{R}_\alpha+\boldsymbol{r}_\alpha) = \left(\frac{a_\alpha}{r_\alpha}\right)^3(\boldsymbol{\Omega}_\alpha\times\boldsymbol{r}_\alpha), \tag{5.76}$$

$$p^{(S,t,0)ind}_\alpha(\boldsymbol{R}_\alpha+\boldsymbol{r}_\alpha) = \frac{3\eta}{2}\frac{a_\alpha}{r_\alpha^2}\,\boldsymbol{n}_\alpha\cdot\boldsymbol{U}^{(t)}_\alpha. \tag{5.77}$$

Relations (5.75) and (5.77) coincide with the well-known distributions of the fluid velocity and pressure induced by a sphere moving with the constant velocity $\boldsymbol{U}_\alpha$ in the immovable fluid ($\boldsymbol{v}^{inf} = 0$) or with the velocity and pressure fields of the fluid uniformly moving with the velocity $\boldsymbol{v}^{inf}$ relative to an immovable sphere ($\boldsymbol{U}_\alpha = 0$) [5]. Relation (5.76) describes the known distribution of the velocity of the fluid induced by a sphere rotating in it with the constant angular velocity $\boldsymbol{\Omega}_\alpha$ [5].

Using relations (5.68), (5.69), and (5.71), we obtain the following main contributions to the fluid velocity and pressure induced by particle β in the vicinity of the surface of particle α for $r_\alpha \ll R_{\alpha\beta}$ (the near zone):

$$\boldsymbol{v}_\beta^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3}{4}\sigma_{\beta\alpha}\left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right)\cdot \boldsymbol{U}_\beta^{(t)}, \tag{5.78}$$

$$\boldsymbol{v}_\beta^{(S,r,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx a_\beta \sigma_{\beta\alpha}^2\left(\boldsymbol{\Omega}_\beta \times \boldsymbol{n}_{\alpha\beta}\right), \tag{5.79}$$

$$p_\beta^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3}{2}\eta\frac{\sigma_{\beta\alpha}}{R_{\alpha\beta}}\,\boldsymbol{n}_{\alpha\beta}\cdot \boldsymbol{U}_\beta^{(t)}. \tag{5.80}$$

However, for the correct description of the fluid velocity and pressure up to these powers of the parameter σ, these quantities should be also found in the first approximation.

Using relations (5.5), (5.6), and (3.55), we obtain the velocity and pressure fields of the fluid induced by particle $\beta \neq \alpha$ in the far zone, i.e., for $r_\alpha \gg R_{\alpha\beta}$. In zero approximation with respect to the ratio $R_{\alpha\beta}/r_\alpha$, we obtain the following space distributions of the fluid velocity and pressure induced by a system of N spheres far from it:

$$\boldsymbol{v}^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3}{4r_\alpha}\left(\boldsymbol{I} + \boldsymbol{n}_\alpha\boldsymbol{n}_\alpha\right)\cdot \sum_{\beta=1}^{N} a_\beta \boldsymbol{U}_\beta^{(t)}, \tag{5.81}$$

$$\boldsymbol{v}^{(S,r,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{1}{r_\alpha^2}\sum_{\beta=1}^{N} a_\beta^3\left(\boldsymbol{\Omega}_\beta \times \boldsymbol{n}_\alpha\right), \tag{5.82}$$

$$p^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3\eta}{2}\frac{1}{r_\alpha^2}\sum_{\beta=1}^{N} a_\beta\,\boldsymbol{n}_\alpha\cdot \boldsymbol{U}_\beta^{(t)}. \tag{5.83}$$

In this approximation, r_α can be interpreted as the distance from a certain point inside this system (e.g., its center) to the point of observation.

Naturally, these relations can be also derived by using the superposition principle for the velocity and pressure fields of the fluid induced by noninteracting particles keeping only the main terms in them.

In the case of equal particles $a_\beta = a$, relations (5.81)–(5.83) can be represented as follows:

$$\boldsymbol{v}^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3}{4}\frac{a}{r_\alpha}\left(\boldsymbol{I} + \boldsymbol{n}_\alpha \boldsymbol{n}_\alpha\right) \cdot \boldsymbol{U}^{tot}, \tag{5.84}$$

$$\boldsymbol{v}^{(S,r,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{a^3}{r_\alpha^2}\left(\boldsymbol{\Omega}^{tot} \times \boldsymbol{n}_\alpha\right), \tag{5.85}$$

$$p^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3\eta}{2}\frac{a}{r_\alpha^2}\, \boldsymbol{n}_\alpha \cdot \boldsymbol{U}^{tot}, \tag{5.86}$$

where

$$\boldsymbol{U}^{tot} = \sum_{\beta=1}^{N} \boldsymbol{U}_\beta^{(t)}, \tag{5.87}$$

$$\boldsymbol{\Omega}^{tot} = \sum_{\beta=1}^{N} \boldsymbol{\Omega}_\beta. \tag{5.88}$$

Thus, the velocity and pressure fields of the fluid induced by the system of equal spheres in the far zone, in zero approximation with respect to the ratio of the typical distance between two spheres to the distance to the point of observation, can represented as the velocity and pressure fields of the fluid induced by a single sphere moving in the fluid with the translational velocity $\boldsymbol{U}^{tot} + \boldsymbol{v}^{inf}$ and rotating with the angular velocity $\boldsymbol{\Omega}^{tot}$. In the particular case of two equal spheres rotating in the opposite directions with equal angular velocities, their rotation has no effect on the fluid velocity far from the spheres within the framework of the considered approximation.

If all spheres move with the same translational velocity $\boldsymbol{U}_\beta = \boldsymbol{U}_0$ ($\boldsymbol{U}_\beta^{(t)} \equiv \boldsymbol{U}^{(t)} = \boldsymbol{U}_0 - \boldsymbol{v}^{inf}$), then

$$\boldsymbol{v}^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3}{4}\frac{a_{tot}}{r_\alpha}\left(\boldsymbol{I} + \boldsymbol{n}_\alpha \boldsymbol{n}_\alpha\right) \cdot \boldsymbol{U}^{(t)}, \tag{5.89}$$

$$p^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3\eta}{2}\frac{a_{tot}}{r_\alpha^2}\, \boldsymbol{n}_\alpha \cdot \boldsymbol{U}^{(t)}, \tag{5.90}$$

where $a_{tot} = \sum_{\beta=1}^{N} a_\beta$. In this case, the action of the system of spheres in the far zone is equivalent to the action of a single sphere of radius a_{tot}.

If all spheres rotate with the same angular velocity $\boldsymbol{\Omega}_\beta = \boldsymbol{\Omega}$, then we obtain the following relation for the fluid velocity:

$$\boldsymbol{v}^{(S,r,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{3}{4\pi}\frac{V_{tot}}{r_\alpha^2}\,(\boldsymbol{\Omega}\times\boldsymbol{n}_\alpha)\,, \tag{5.91}$$

which corresponds to the fluid velocity induced by a single sphere occupying the volume $V_{tot} = \sum\limits_{\beta=1}^{N} V_\beta = (4\pi/3)\sum\limits_{\beta=1}^{N} a_\beta^3$ far from it.

According to (5.22), (5.27), (5.29), (5.20), and (5.33), in the approximation of noninteracting particles, we obtain the following relations for the components of the force acting by the fluid on particle α caused by translational and rotational motions of particles and the external force field:

$$\boldsymbol{F}_\alpha^{(t,0)} = -\xi_\alpha \boldsymbol{U}_\alpha^{(t)}, \tag{5.92}$$

$$\boldsymbol{F}_\alpha^{(r,0)} = 0, \tag{5.93}$$

$$\boldsymbol{F}_\alpha^{(ext,0)} = \xi_\alpha \overline{\boldsymbol{v}^{(0)sol}(\boldsymbol{r})}^{S_\alpha} - \tilde{\boldsymbol{F}}_\alpha^{(sol)ext}. \tag{5.94}$$

Relation (5.92) is the classical Stokes force acting by the fluid on a sphere moving in it with the constant velocity $\boldsymbol{U}_\alpha$ if the fluid moves with constant velocity $\boldsymbol{v}^{inf}$. Relation (5.93) illustrates the well-known fact of the absence of a force acting on a sphere due to its rotation. The first term in relation (5.94) corresponds to the known Faxén relation [1] defined by the classical Stokes law with the velocity equal to the velocity of inhomogeneous motion of the fluid in the absence of spheres averaged over the surface of sphere α.

Substituting (5.92)–(5.94) into (2.27) and taking into account the absence of the inertial force $\boldsymbol{F}_\alpha^{in}$ defined by (2.33) in the stationary case, we obtain the following relation for the total force acting on particle α:

$$\boldsymbol{F}_\alpha^{tot} = \boldsymbol{F}_\alpha^{ext} - \tilde{\boldsymbol{F}}_\alpha^{ext} + \boldsymbol{F}_\alpha^{(t,0)} + \xi_\alpha \overline{\boldsymbol{v}^{(0)sol}(\boldsymbol{r})}^{S_\alpha}, \tag{5.95}$$

where

$$\tilde{\boldsymbol{F}}_\alpha^{ext} = \tilde{\boldsymbol{F}}_\alpha^{(p)ext} + \tilde{\boldsymbol{F}}_\alpha^{(sol)ext} = \int\limits_{V_\alpha} d\boldsymbol{r}\, \boldsymbol{F}^{ext}(\boldsymbol{r}) \tag{5.96}$$

is the force acting by the external force field $\boldsymbol{F}^{ext}(\boldsymbol{r})$ on the fluid occupying the volume V_α. The difference $\boldsymbol{F}_\alpha^{ext} - \tilde{\boldsymbol{F}}_\alpha^{ext}$ in (5.95) is the force acting on particle α in an arbitrary force field $\boldsymbol{F}^{ext}(\boldsymbol{r})$ corrected for the buoyancy force for this force field.

To determine the torque $\boldsymbol{T}_\alpha^{(0)}$ exerted by the fluid on sphere α, we substitute relations (5.52)–(5.54) into (5.23) and perform certain transformations taking into account the validity of the following relation:

$$\sum_{m=-1}^{1} \left(\boldsymbol{e}_m \times \boldsymbol{b}_m^T\right) = -\sum_{m=-1}^{1} \left(\boldsymbol{e}_m \times \boldsymbol{b}_m\right) \tag{5.97}$$

for any vectors b_m and b_m^T defined by relations (5.55) and (5.57). As a result, we get

$$\boldsymbol{T}_\alpha^{(t,0)} = 0, \tag{5.98}$$

$$\boldsymbol{T}_\alpha^{(r,0)} = -\xi_\alpha^R \boldsymbol{\Omega}_\alpha, \tag{5.99}$$

$$\boldsymbol{T}_\alpha^{(ext,0)} = 2\xi_\alpha \overline{\left(\boldsymbol{a}_\alpha \times \boldsymbol{v}^{(0)sol}(\boldsymbol{r})\right)}^{S_\alpha} - \tilde{\boldsymbol{T}}_\alpha^{(sol)ext}. \tag{5.100}$$

According to (5.98), we have the well-known result of the absence of the torque acting on a sphere moving with a constant velocity in an unbounded fluid. Relation (5.99) is the classical result for the torque exerted by the fluid on a sphere rotating in it with constant angular velocity $\boldsymbol{\Omega}_\alpha$ [1]. The first term in (5.100) corresponds to the torque exerted by the fluid on sphere α due to the inhomogeneous motion of the fluid in the absence of particles caused by the solenoidal component of the external force field. For a homogeneous distribution of the fluid velocity, this torque is equal to zero. The second term in (5.100) defined as follows:

$$\tilde{\boldsymbol{T}}_\alpha^{(sol)ext} = \int_{V_\alpha} d\boldsymbol{r} \, \left(\boldsymbol{r}_\alpha \times \boldsymbol{F}^{(sol)ext}(\boldsymbol{r})\right) \tag{5.101}$$

is the torque exerted on the fluid sphere occupying the volume V_α due to the solenoidal component $\boldsymbol{F}^{(sol)ext}(\boldsymbol{r})$ of the external force field $\boldsymbol{F}^{ext}(\boldsymbol{r})$.

Substituting (5.98)–(5.100) into (2.28), we obtain the following relation for the total torque exerted on sphere α:

$$\boldsymbol{T}_\alpha^{tot} = \boldsymbol{T}_\alpha^{ext} - \tilde{\boldsymbol{T}}_\alpha^{ext} + \boldsymbol{T}_\alpha^{(r,0)} + 2\xi_\alpha \overline{\left(\boldsymbol{a}_\alpha \times \boldsymbol{v}^{(0)sol}(\boldsymbol{r})\right)}^{S_\alpha}, \tag{5.102}$$

where

$$\tilde{\boldsymbol{T}}_\alpha^{ext} = \tilde{\boldsymbol{T}}_\alpha^{(p)ext} + \tilde{\boldsymbol{T}}_\alpha^{(sol)ext} = \int_{V_\alpha} d\boldsymbol{r} \, \left(\boldsymbol{r}_\alpha \times \boldsymbol{F}^{ext}(\boldsymbol{r})\right) \tag{5.103}$$

is the torque acting due to the external force field $\boldsymbol{F}^{ext}(\boldsymbol{r})$ on the fluid occupying the volume V_α. The difference $\boldsymbol{T}_\alpha^{ext} - \tilde{\boldsymbol{T}}_\alpha^{ext}$ in (5.102) is the torque acting on sphere α in an arbitrary force field $\boldsymbol{F}^{ext}(\boldsymbol{r})$ corrected for the buoyancy torque for this force field.

Thus, the external force field makes the contribution to the total force and torque exerted on sphere α both due to the buoyancy force $\tilde{\boldsymbol{F}}_\alpha^{ext}$ and torque $\tilde{\boldsymbol{T}}_\alpha^{ext}$ on the one hand, and due to the inhomogeneous distribution of the fluid velocity in this force field in the absence of particles, on the other.

B. $n = 1$. The First Iteration

For the first iteration, with regard for (5.24), the system of equations (5.14) is reduced to the form

$$\sum_{m_2=-l_1}^{l_1} \boldsymbol{T}_{\alpha,l_1m_1}^{\alpha,l_1m_2} \cdot \boldsymbol{f}_{\alpha,l_1m_2}^{(1)} = -\sum_{\beta\neq\alpha} \left\{ \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}_{\alpha,l_1m_1}^{\beta,l_2m_2} \cdot \boldsymbol{f}_{\beta,l_2m_2}^{(0)} + \boldsymbol{v}_{\alpha,l_1m_1}^{\beta(sol)} \right\}, \tag{5.104}$$

where the quantities $\boldsymbol{f}_{\alpha,l_2m_2}^{(0)}$ are determined by relations (5.27), (5.29), (5.30) for $l_2 = 0$, (5.52)–(5.54) for $l_2 = 1$, and (5.62) for $l_2 \geq 2$.

Taking into account relations (3.34), (3.35), (3.37), (5.3), and (5.4), we represent the solution of system (5.104) for $l_1 = 0$ in the form (5.32) with $n = 1$, where

$$\boldsymbol{f}_{\alpha,00}^{(t,1)} = -\xi_\alpha \sum_{\beta\neq\alpha} \xi_\beta \boldsymbol{T}_M(\boldsymbol{R}_{\alpha\beta}) \cdot \boldsymbol{U}_\beta^{(t)}, \tag{5.105}$$

$$\boldsymbol{f}_{\alpha,00}^{(r,1)} = \sum_{\beta\neq\alpha} \xi_\beta a_\beta \sigma_{\alpha\beta}\sigma_{\beta\alpha} \left(\boldsymbol{n}_{\alpha\beta} \times \boldsymbol{\Omega}_\beta\right), \tag{5.106}$$

$$\begin{aligned} \boldsymbol{f}_{\alpha,00}^{(ext,1)} &= -\xi_\alpha \sum_{\beta\neq\alpha} \Big\{ \boldsymbol{v}_{\alpha,00}^{\beta(sol)} + \boldsymbol{T}_M(\boldsymbol{R}_{\alpha\beta}) \cdot \left(\tilde{\boldsymbol{F}}_\beta^{(sol)ext} - \xi_\beta \overline{\boldsymbol{v}^{(0)sol}(\boldsymbol{r})}^{S_\beta} \right) \\ &\quad + \sum_{l_2=1}^{\infty} \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}_{\alpha,00}^{\beta,l_2m_2} \cdot \boldsymbol{f}_{\beta,l_2m_2}^{(ext,0)} \Big\}, \end{aligned} \tag{5.107}$$

where

$$\begin{aligned} \boldsymbol{T}_M(\boldsymbol{R}_{\alpha\beta}) \equiv \boldsymbol{T}_{\alpha,00}^{\beta,00}(a_\alpha, a_\beta, \boldsymbol{R}_{\alpha\beta}) &= \frac{1}{8\pi\eta R_{\alpha\beta}} \Big\{ (\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}) \\ &\quad + \left(\sigma_{\alpha\beta}^2 + \sigma_{\beta\alpha}^2\right) \left(\frac{1}{3}\boldsymbol{I} - \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta} \right) \Big\} \end{aligned} \tag{5.108}$$

is the modified Oseen tensor [30].

According to (5.105)–(5.107), in this approximation, $\boldsymbol{f}^{(r,1)}_{\alpha,00} \sim \sigma^2$, $\boldsymbol{f}^{(t,1)}_{\alpha,00}$ contains terms proportional to σ and σ^3, and $\boldsymbol{f}^{(ext,1)}_{\alpha,00}$ has terms $\sim 1/R^n_{\alpha\beta}$, where $n = 1, 2, \ldots$, furthermore,

$$\boldsymbol{f}^{(ext,1)}_{\alpha,00} \approx \frac{3}{4}\xi_\alpha \sum_{\beta\neq\alpha} \xi_\beta \sigma_{\beta\alpha} \left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right) \cdot \overline{\boldsymbol{v}^{(0)sol}(\boldsymbol{r})}^{S_\beta} + o\left(\frac{1}{R^2_{\alpha\beta}}\right). \tag{5.109}$$

For $l_1 = 1$, system (5.104) has the form

$$\sum_{m_2=-1}^{1} \boldsymbol{K}^{1m_2}_{1m_1,00} \cdot \boldsymbol{f}^{(1)}_{\alpha,1m_2} = -\frac{\xi_\alpha}{\sqrt{\pi}} \sum_{\beta\neq\alpha} \left\{ \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}^{\beta,l_2m_2}_{\alpha,1m_1} \cdot \boldsymbol{f}^{(0)}_{\beta,l_2m_2} + \boldsymbol{v}^{\beta(sol)}_{\alpha,1m_1} \right\}, \tag{5.110}$$

which differs from analogous system (5.34) for the iteration of noninteracting particles only by the right-hand side. This implies that system (5.110) is consistent only if

$$\sum_{m_1=-1}^{1} \boldsymbol{e}_{m_1} \cdot \sum_{\beta\neq\alpha} \left\{ \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}^{\beta,l_2m_2}_{\alpha,1m_1} \cdot \boldsymbol{f}^{(0)}_{\beta,l_2m_2} + \boldsymbol{v}^{\beta(sol)}_{\alpha,1m_1} \right\} = 0. \tag{5.111}$$

This condition is satisfied because the following relations are true:

$$\sum_{m_1=-1}^{1} \boldsymbol{e}_{m_1} \cdot \boldsymbol{T}^{\beta,l_2m_2}_{\alpha,1m_1} = 0, \tag{5.112}$$

$$\sum_{m_1=-1}^{1} \boldsymbol{e}_{m_1} \cdot \boldsymbol{v}^{\beta(sol)}_{\alpha,1m_1} = 0\,. \tag{5.113}$$

Therefore, system (5.110) has an infinite number of solutions that differ only in the potential component. To determine the unique solution, it is necessary to introduce an additional equation linearly independent of the equations of this system. To derive this equation, we use relations (5.47)–(5.50) and get

$$Y^{(1)}_\alpha \equiv \sum_{m=-1}^{1} \boldsymbol{e}_m \cdot \boldsymbol{f}^{(1)}_{\alpha,1m} = 4\pi\sqrt{3}\, a^2_\alpha \sum_{\beta\neq\alpha} \left\{ p^{(V)ind}_\beta(\boldsymbol{R}_\alpha) + p^{(S,0)ind}_\beta(\boldsymbol{R}_\alpha) \right\}, \quad \alpha = 1, 2, \ldots, N, \tag{5.114}$$

where $p^{(S,0)ind}_\beta(\boldsymbol{R}_\alpha)$ is defined by relation (5.50) with $\boldsymbol{f}_{\beta,l_2m_2} \to \boldsymbol{f}^{(0)}_{\beta,l_2m_2}$.

Unlike Eq. (5.48), Eq. (114) contains the nonzero right-hand side. Therefore, the interaction between the particles leads to the appearance of the potential components $\boldsymbol{f}^{(p)}_\alpha(\boldsymbol{a})$ of the induced surface forces densities $\boldsymbol{f}_\alpha(\boldsymbol{a})$.

We solve the system of equation (5.110), (5.114) in a similar way as in the case of noninteracting particles. The final result has the form

$$\boldsymbol{f}^{(1)}_{\alpha,1m} = \boldsymbol{f}^{(t,1)}_{\alpha,1m} + \boldsymbol{f}^{(r,1)}_{\alpha,1m} + \boldsymbol{f}^{(ext,1)}_{\alpha,1m} + \boldsymbol{f}^{(p,1)}_{\alpha,1m}, \tag{5.115}$$

where

$$\boldsymbol{f}^{(\epsilon,1)}_{\alpha,1m} = -\frac{2}{3}\,\xi_\alpha \left\{ 4\boldsymbol{b}^{(\epsilon,1)}_{\alpha,m} + \left(\boldsymbol{b}^{(\epsilon,1)}_{\alpha,m}\right)^T \right\}, \qquad \epsilon = t, r, ext, \tag{5.116}$$

$$\boldsymbol{f}^{(p,1)}_{\alpha,1m} = \boldsymbol{e}^*_m \frac{Y^{(1)}_\alpha}{3}, \tag{5.117}$$

$$\boldsymbol{b}^{(t,1)}_{\alpha,m_1} = \sum_{\beta\neq\alpha} \sum_{l_2 m_2} \boldsymbol{T}^{\beta,l_2 m_2}_{\alpha,1m_1} \cdot \boldsymbol{f}^{(t,0)}_{\beta,l_2 m_2} = \sum_{\beta\neq\alpha} \xi_\beta \boldsymbol{T}^{\beta,00}_{\alpha,1m_1} \cdot \boldsymbol{U}^{(t)}_\beta, \tag{5.118}$$

$$\begin{aligned}\boldsymbol{b}^{(r,1)}_{\alpha,m_1} &= \sum_{\beta\neq\alpha} \sum_{l_2 m_2} \boldsymbol{T}^{\beta,l_2 m_2}_{\alpha,1m_1} \cdot \boldsymbol{f}^{(r,0)}_{\beta,l_2 m_2} \\ &= \frac{2}{3\eta} \sum_{\beta\neq\alpha} \xi_\beta \sigma_{\alpha\beta} \sigma^2_{\beta\alpha} \sum_{m=-2}^{2} \left(\boldsymbol{\Omega}_\beta \times \boldsymbol{W}^{00}_{1m_1,2m}\right) Y_{2m}(\Theta_{\alpha\beta}, \Phi_{\alpha\beta}),\end{aligned} \tag{5.119}$$

$$\boldsymbol{b}^{(ext,1)}_{\alpha,m_1} = \sum_{\beta\neq\alpha} \left\{ \sum_{l_2 m_2} \boldsymbol{T}^{\beta,l_2 m_2}_{\alpha,1m_1} \cdot \boldsymbol{f}^{(ext,0)}_{\beta,l_2 m_2} + \boldsymbol{v}^{\beta(sol)}_{\alpha,1m_1} \right\}, \tag{5.120}$$

and the quantities $\left(\boldsymbol{b}^{(\epsilon,1)}_{\alpha,m}\right)^T$ are determined by relations (5.56), (5.57), and (5.118)–(5.120).

In the first iteration, we obtain that $\boldsymbol{f}^{(r,1)}_{\alpha,1m} \sim \sigma^3$, $\boldsymbol{f}^{(t,1)}_{\alpha,1m}$ has terms proportional to σ^2 and σ^4, and $\boldsymbol{f}^{(ext,1)}_{\alpha,1m}$ contains terms $\sim 1/R^n_{\alpha\beta}$, where $n = 2, 3, \ldots$.

For $l_1 \geq 2$, the solution of the system of equations (5.104) can be represented in the form (5.32) with $n = 1$, where

$$\boldsymbol{f}^{(t,1)}_{\alpha,l_1 m_1} = -\sum_{\beta\neq\alpha} \xi_\beta \sum_{m_2=-l_1}^{l_1} \tilde{\boldsymbol{T}}^{\alpha,l_1 m_2}_{\alpha,l_1 m_1} \cdot \boldsymbol{T}^{\beta,00}_{\alpha,l_1 m_2} \cdot \boldsymbol{U}^{(t)}_\beta, \tag{5.121}$$

$$\begin{aligned}\boldsymbol{f}^{(r,1)}_{\alpha,l_1 m_1} &= -\frac{\sqrt{3}}{2} \sum_{\beta\neq\alpha} \frac{\xi^R_\beta}{a_\beta} \sum_{m_3=-l_1}^{l_1} \sum_{m_2=-1}^{1} \tilde{\boldsymbol{T}}^{\alpha,l_1 m_3}_{\alpha,l_1 m_1} \cdot \boldsymbol{T}^{\beta,1m_2}_{\alpha,l_1 m_3} \cdot \left(\boldsymbol{\Omega}_\beta \times \boldsymbol{e}^*_{m_2}\right) \\ &= -\sqrt{\pi}(2l_1+1) \sum_{\beta\neq\alpha} \frac{\xi^R_\beta}{a_\beta}\, \sigma^{l_1+1}_{\alpha\beta} \sigma_{\beta\alpha} \sum_{m_2=-l_1}^{l_1} \sum_{m=-(l_1+1)}^{l_1+1} \tilde{\boldsymbol{K}}^{l_1 m_2}_{l_1 m_1,00} \\ &\cdot \left(\boldsymbol{\Omega}_\beta \times \boldsymbol{W}^{00}_{l_1 m_2, l_1+1,m}\right) Y_{l_1+1,m}(\Theta_{\alpha\beta}, \Phi_{\alpha,\beta}),\end{aligned} \tag{5.122}$$

$$\begin{aligned}\boldsymbol{f}^{(ext,1)}_{\alpha,l_1 m_1} &= \sum_{\beta\neq\alpha} \sum_{m_3=-l_1}^{l_1} \tilde{\boldsymbol{T}}^{\alpha,l_1 m_3}_{\alpha,l_1 m_1} \cdot \left\{ \boldsymbol{T}^{\beta,00}_{\alpha,l_1 m_3} \cdot \left\{ \xi_\beta \overline{\boldsymbol{v}^{(0)sol}(\boldsymbol{r})}^{S_\alpha} - \tilde{\boldsymbol{F}}^{(sol)ext}_\beta \right\} + \frac{2}{3}\,\xi_\beta \sum_{m_2=-1}^{1} \boldsymbol{T}^{\beta,1m_2}_{\alpha,l_1 m_3} \right. \\ &\left. \cdot \left\{ 4\boldsymbol{b}^{(ext,1)}_{\beta,m_2} + \left(\boldsymbol{b}^{(ext,1)}_{\beta,m_2}\right)^T \right\} - \sum_{l_2=2}^{\infty} \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}^{\beta,l_2 m_2}_{\alpha,l_1 m_3} \cdot \boldsymbol{f}^{(ext,0)}_{\beta,l_2 m_2} - \boldsymbol{v}^{\beta(sol)}_{\alpha,l_1 m_3} \right\}.\end{aligned} \tag{5.123}$$

Thus, the interaction of particles leads to the appearance of all harmonics for the components of the induced surface force density connected with the translational motion of particles and their rotation.

Since the potential component $\boldsymbol{f}_\alpha^{(p,1)}(\boldsymbol{a}_\alpha)$ of the induced surface force density $\boldsymbol{f}_\alpha^{(1)}(\boldsymbol{a}_\alpha)$ has only harmonics with $l = 1$ defined by (5.117), we obtain the following results for the corresponding fluid velocity and pressure ($r_\alpha \geq a_\alpha$) induced by these potential forces:

$$\boldsymbol{v}_\beta^{(S,p,1)ind}(\boldsymbol{r}) = 0, \qquad \beta = 1, 2, \ldots, N, \tag{5.124}$$

$$p_\beta^{(S,p,1)ind}(\boldsymbol{r}) = \delta_{\beta,\alpha}\,\delta_{r_\alpha,a_\alpha}\,p_\alpha^{(S,p,1)ind}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha + 0), \quad \beta = 1, 2, \ldots, N, \tag{5.125}$$

where

$$p_\alpha^{(S,p,1)ind}(\boldsymbol{R}_\alpha + \boldsymbol{a}_\alpha + 0) = -\frac{Y_\alpha^{(1)}}{4\pi\sqrt{3}\,a_\alpha^2}. \tag{5.126}$$

Thus, the potential component $\boldsymbol{f}_\beta^{(p,1)}(\boldsymbol{a}_\beta)$, where $\beta = 1, 2, \ldots N$, of the induced surface force density makes no contribution to the fluid velocity, while the fluid pressure caused by this potential force is equal to zero everywhere with the exception of the surface of the sphere where this force is distributed.

In view of (5.117), the potential component $\boldsymbol{f}_\alpha^{(p,1)}(\boldsymbol{a}_\alpha)$ is represented in the form

$$\boldsymbol{f}_\alpha^{(p,1)} = \frac{\boldsymbol{a}_\alpha}{a_\alpha}\frac{Y_\alpha^{(1)}}{4\pi\sqrt{3}}, \tag{5.127}$$

and, hence, has only the radial constant component. In this case, it is easy to see that the potential component of the induced surface force has no effect on the forces and torques exerted by the fluid on the particles immersed in it.

These conclusions concerning the influence of the potential component of the induced surface force densities on the fluid velocity and pressure as well as on the forces and torques exerted by the fluid on particles also remain valid for any nth iteration if the potential component $\boldsymbol{f}_\alpha^{(p,n)}(\boldsymbol{a}_\alpha)$ of the induced surface force density $\boldsymbol{f}_\alpha^{(n)}(\boldsymbol{a}_\alpha)$ corresponding to this iteration has only harmonics with $l = 1$ defined by a relation similar to (5.117).

By virtue of relations (5.105), (5.106), (5.115), (5.116), (5.118), (5.119), (5.121), and (5.122), in this approximation, the main contribution to the fluid velocity and pressure in the vicinity of particle α due to the interaction between the particles caused by the motion of particles and their rotation is given by the quantities $\boldsymbol{v}_\alpha^{(S,t,1)ind}(\boldsymbol{r})$, $\boldsymbol{v}_\alpha^{(S,r,1)ind}(\boldsymbol{r})$, $p_\alpha^{(S,t,1)ind}(\boldsymbol{r})$, and $p_\alpha^{(S,r,1)ind}(\boldsymbol{r})$, moreover, $\boldsymbol{v}_\alpha^{(S,t,1)ind}(\boldsymbol{r})$ and $\boldsymbol{v}_\alpha^{(S,r,1)ind}(\boldsymbol{r})$ are of the same order in the parameter σ that the quantities $\boldsymbol{v}_\beta^{(S,t,0)ind}(\boldsymbol{r})$ and $\boldsymbol{v}_\beta^{(S,r,0)ind}(\boldsymbol{r})$, where $\beta \neq \alpha$, defined, respectively, by relations (5.78) and (5.79) in the approximation of noninteracting particles. Near the surface of particle α, for $r_\alpha \ll R_{\alpha\beta}$, the main terms for the fluid velocity and pressure due to the hydrodynamic interaction between the particles can be written as follows:

$$\sum_{\beta\neq\alpha} \boldsymbol{v}_\beta^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) + \boldsymbol{v}_\alpha^{(S,t,1)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx \frac{9}{8}\left(1 - \frac{a_\alpha}{r_\alpha}\right)(\boldsymbol{I} - \boldsymbol{n}_\alpha\boldsymbol{n}_\alpha)\cdot \sum_{\beta\neq\alpha} \sigma_{\beta\alpha}\,(\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta})\cdot \boldsymbol{U}_\beta^{(t)}, \tag{5.128}$$

$$\sum_{\beta\neq\alpha} \boldsymbol{v}_\beta^{(S,r,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) + \boldsymbol{v}_\alpha^{(S,r,1)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx -\frac{3}{2}\left(1 - \frac{a_\alpha}{r_\alpha}\right)(\boldsymbol{I} - 3\boldsymbol{n}_\alpha\boldsymbol{n}_\alpha)\cdot \sum_{\beta\neq\alpha} a_\beta \times \sigma_{\beta\alpha}^2\,(\boldsymbol{n}_{\alpha\beta}\times\boldsymbol{\Omega}_\beta)\,, \tag{5.129}$$

$$\sum_{\beta\neq\alpha} p_\beta^{(S,t,0)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) + p_\alpha^{(S,t,1)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx p_\alpha^{(S,t,1)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) \approx -\left(1 - \delta_{r_\alpha,a_\alpha}\frac{1}{2}\right)\boldsymbol{n}_\alpha \cdot \frac{9\eta}{8r_\alpha}\sum_{\beta\neq\alpha}\sigma_{\beta\alpha}\,(\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta})\cdot\boldsymbol{U}_\beta^{(t)}, \tag{5.130}$$

$$p_\alpha^{(S,r,1)ind}(\boldsymbol{R}_\alpha + \boldsymbol{r}_\alpha) = \left(1 - \delta_{r_\alpha,a_\alpha}\frac{1}{2}\right)\frac{3\eta}{2r_\alpha}\,\boldsymbol{n}_\alpha\cdot\sum_{\beta\neq\alpha} a_\beta\sigma_{\beta\alpha}^2\,(\boldsymbol{n}_{\alpha\beta}\times\boldsymbol{\Omega}_\beta)\,. \tag{5.131}$$

According to (5.128)–(5.131), the main terms of the fluid velocity and pressure caused by the hydrodynamic interaction between the particles due to their translational motion and rotation are proportional to the first and second powers of the parameter σ, respectively.

Substituting (5.105) and (5.106) into (5.22) and (5.116), (5.118), and (5.119) into (5.23) and taking relation (5.97) into account, we obtain the following relations for the forces and torques exerted by the fluid on sphere α for this iteration corresponding to the translational motion of particles and their rotation:

$$\boldsymbol{F}_\alpha^{(t,1)} = \xi_\alpha \sum_{\beta\neq\alpha} \xi_\beta \boldsymbol{T}_M(\boldsymbol{R}_{\alpha\beta})\cdot\boldsymbol{U}_\beta^{(t)}, \tag{5.132}$$

$$\boldsymbol{F}_\alpha^{(r,1)} = -\sum_{\beta\neq\alpha} \xi_\beta a_\beta \sigma_{\alpha\beta}\sigma_{\beta\alpha}\,(\boldsymbol{n}_{\alpha\beta}\times\boldsymbol{\Omega}_\beta)\,, \tag{5.133}$$

$$\boldsymbol{T}_\alpha^{(t,1)} = -\xi_\alpha a_\alpha \sum_{\beta\neq\alpha} \sigma_{\alpha\beta}\sigma_{\beta\alpha} \left(\boldsymbol{n}_{\alpha\beta} \times \boldsymbol{U}_\beta^{(t)}\right), \tag{5.134}$$

$$\boldsymbol{T}_\alpha^{(r,1)} = -\frac{\xi_\alpha^R}{2} \sum_{\beta\neq\alpha} \sigma_{\beta\alpha}^3 \left(\boldsymbol{I} - 3\boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right) \cdot \boldsymbol{\Omega}_\beta. \tag{5.135}$$

Relation (5.132) agrees with the second term in relation (44) in [30] corresponding to the first iteration. It is worth noting that relation (5.133) for the force exerted by the fluid on a sphere due to rotation of the rest spheres coincides with the known result for two spheres [12], while the corresponding force given by the first term in relation (49) in [30] is proportional to the inverse tensor $\tilde{\boldsymbol{K}}_{1m_1,00}^{1m_2}$, which, as was mentioned above, does not exist. Relations (5.134) and (5.135) also agree with the known relations for the torques exerted by the fluid on a sphere due to the translational motion and rotation of the rest spheres [1,12].

According to (5.132) and (5.108), the force $\boldsymbol{F}_\alpha^{(t,1)}$ contains terms proportional to the first and third powers of the dimensionless parameter σ. Restricting ourselves only to the terms proportional to σ, we obtain

$$\boldsymbol{F}_\alpha^{(t,1)} \approx \frac{3}{4}\xi_\alpha \sum_{\beta\neq\alpha} \sigma_{\beta\alpha} \left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right) \cdot \boldsymbol{U}_\beta^{(t)}. \tag{5.136}$$

The terms proportional to σ^3 in (5.132) should be retained only if the problem is solved within the framework of at least the third iteration.

Often, it is necessary to determine the velocities of particles in a fluid in given external force fields (in particular, the sedimentation velocity of particles in a fluid due to the action of the gravity force) and to investigate the influence of hydrodynamic interactions between the particles on their motion. Let $\boldsymbol{U}_\alpha$ and $\boldsymbol{\Omega}_\alpha$ be, respectively, the velocity of translational motion and the angular velocity of a single sphere α in a fluid caused by the action of a certain external force field. In the case of several spheres, these velocities change due to hydrodynamic interactions between the spheres. Using relations (5.133)–(5.136), in this approximation, we can represent the changed velocities $\tilde{\boldsymbol{U}}_\alpha$ and $\tilde{\boldsymbol{\Omega}}_\alpha$ as follows:

$$\tilde{\boldsymbol{U}}_\alpha = \boldsymbol{U}_\alpha + \sum_{\beta\neq\alpha} \boldsymbol{U}_{\alpha\beta}^{(1)}, \tag{5.137}$$

$$\tilde{\boldsymbol{\Omega}}_\alpha = \boldsymbol{\Omega}_\alpha + \sum_{\beta\neq\alpha} \boldsymbol{\Omega}_{\alpha\beta}^{(1)}, \tag{5.138}$$

where $\boldsymbol{U}^{(1)}_{\alpha\beta}$ and $\boldsymbol{\Omega}^{(1)}_{\alpha\beta}$ are, respectively, the changes in the velocity of translational motion and the angular velocity of sphere α due to the motion and rotation of sphere β. Up to the main terms in powers of the parameter σ, these quantities have the form

$$\boldsymbol{U}^{(1)}_{\alpha\beta} = \sigma_{\beta\alpha}\left\{\frac{3}{4}\left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right)\cdot\boldsymbol{U}^{(t)}_{\beta} - \sigma_{\beta\alpha}a_{\beta}\left(\boldsymbol{n}_{\alpha\beta}\times\boldsymbol{\Omega}_{\beta}\right)\right\}, \tag{5.139}$$

$$\boldsymbol{\Omega}^{(1)}_{\alpha\beta} = -\frac{3}{2}\sigma_{\beta\alpha}\left\{\frac{1}{2R_{\alpha\beta}}\left(\boldsymbol{n}_{\alpha\beta}\times\boldsymbol{U}^{(t)}_{\beta}\right) + \sigma^{2}_{\beta\alpha}\left(\frac{1}{3}\boldsymbol{I} - \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right)\cdot\boldsymbol{\Omega}_{\beta}\right\}. \tag{5.140}$$

The first term in relation (5.139) agrees with the main term in the relation given in [1] for the problem of sedimentation of two particles with constant velocities without rotation under the action of the gravity force. The first term in relation (5.140) coincides with the well-known result in [1] for the angular velocity of sphere α, whose rotation is induced by the motion of sphere β moving in the fluid with the relative translational velocity $\boldsymbol{U}^{(t)}_{\beta}$.

C. $n = 2$. The Second Iteration

Using Eqs. (5.14), (5.49), and (5.50), we obtain the following systems of equations for the nth iteration, where $n \geq 2$:

$$\sum_{m_2=-l_1}^{l_1}\boldsymbol{T}^{\alpha,l_1m_2}_{\alpha,l_1m_1}\cdot\boldsymbol{f}^{(n)}_{\alpha,l_1m_2} = -\sum_{\beta\neq\alpha}\sum_{l_2m_2}\boldsymbol{T}^{\beta,l_2m_2}_{\alpha,l_1m_1}\cdot\boldsymbol{f}^{(n-1)}_{\beta,l_2m_2}, \tag{5.141}$$

$$Y^{(n)}_{\alpha} \equiv \sum_{m=-1}^{1}\boldsymbol{e}_m\cdot\boldsymbol{f}^{(n)}_{\alpha,1m} = 4\pi\sqrt{3}\,a^{2}_{\alpha}\sum_{\beta\neq\alpha}p^{(S,n-1)ind}_{\beta}(\boldsymbol{R}_{\alpha}), \tag{5.142}$$

where $p^{(S,n-1)ind}_{\beta}(\boldsymbol{R}_{\alpha})$ is defined by relation (5.50) with $\boldsymbol{f}^{(n-1)}_{\beta,lm}$ substituted for $\boldsymbol{f}_{\beta,lm}$.

Putting $n = 2$ in (5.141) and (5.142), we obtain the required equations for the second iteration. Further, putting $l_1 = 0$ and using relations (5.105)–(5.107) obtained for the first iteration, we obtain the following solution:

$$\begin{aligned}\boldsymbol{f}^{(t,2)}_{\alpha,00} = \xi_{\alpha}\sum_{\beta\neq\alpha}\Bigg\{&\xi_{\beta}\boldsymbol{T}_{M}(\boldsymbol{R}_{\alpha\beta})\cdot\sum_{\gamma\neq\beta}\xi_{\gamma}\boldsymbol{T}_{M}(\boldsymbol{R}_{\beta\gamma})\cdot\boldsymbol{U}^{(t)}_{\gamma} + \frac{2}{3}\,\xi_{\beta}\sum_{m_2=-1}^{1}\boldsymbol{T}^{\beta,1m_2}_{\alpha,00}\\ &\cdot\left\{4\boldsymbol{b}^{(t,1)}_{\beta,m_2} + \left(\boldsymbol{b}^{(t,1)}_{\beta,m_2}\right)^{T}\right\} - \sum_{l_2=2}^{\infty}\sum_{m_2=-l_2}^{l_2}\boldsymbol{T}^{\beta,l_2m_2}_{\alpha,00}\cdot\boldsymbol{f}^{(t,1)}_{\beta,l_2m_2}\Bigg\},\end{aligned} \tag{5.143}$$

$$\boldsymbol{f}^{(r,2)}_{\alpha,00} = \xi_\alpha \sum_{\beta\neq\alpha} \Bigg\{ \boldsymbol{T}_M(\boldsymbol{R}_{\alpha\beta}) \cdot \sum_{\gamma\neq\beta} \xi_\gamma a_\gamma \sigma_{\beta\gamma}\sigma_{\gamma\beta} \left(\boldsymbol{\Omega}_\gamma \times \boldsymbol{n}_{\beta\gamma}\right) + \frac{2}{3}\,\xi_\beta \sum_{m_2=-1}^{1} \boldsymbol{T}^{\beta,1m_2}_{\alpha,00}$$

$$\cdot \left\{ 4\boldsymbol{b}^{(r,1)}_{\beta,m_2} + \left(\boldsymbol{b}^{(r,1)}_{\beta,m_2}\right)^T \right\} - \sum_{l_2=2}^{\infty} \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}^{\beta,l_2m_2}_{\alpha,00} \cdot \boldsymbol{f}^{(r,1)}_{\beta,l_2m_2} \Bigg\}, \tag{5.144}$$

$$\boldsymbol{f}^{(ext,2)}_{\alpha,00} = -\xi_\alpha \sum_{\beta\neq\alpha} \sum_{l_2m_2} \boldsymbol{T}^{\beta,l_2m_2}_{\alpha,00} \cdot \boldsymbol{f}^{(ext,1)}_{\beta,l_2m_2}. \tag{5.145}$$

According to (5.143) and (5.144), the quantities $\boldsymbol{f}^{(t,2)}_{\alpha,00}$ and $\boldsymbol{f}^{(r,2)}_{\alpha,00}$ are infinite power series in the parameter σ starting from the second and third powers, respectively, which essentially differ them from the corresponding relations (5.105) and (5.106) obtained for the first iteration. The quantity $\boldsymbol{f}^{(t,2)}_{\alpha,00}$ contains only even powers of σ. We note that the representation for a harmonic of the induced surface force density as an infinite power series in the parameter σ is also valid both for other l ($l = 3, 4, \ldots$) and for higher iterations ($n = 3, 4, \ldots$).

For $l_1 = 1$, the system of equations (5.141) has the form

$$\sum_{m_2=-1}^{1} \boldsymbol{K}^{1m_2}_{1m_1,00} \cdot \boldsymbol{f}^{(2)}_{\alpha,1m_2} = -\frac{\xi_\alpha}{\sqrt{\pi}} \sum_{\beta\neq\alpha} \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}^{\beta,l_2m_2}_{\alpha,1m_1} \cdot \boldsymbol{f}^{(1)}_{\beta,l_2m_2}, \tag{5.146}$$

which is similar to system (5.110). Since the determinant of this system is equal to zero and, by virtue of relation (5.112), this system is consistent, it has an infinite number of solutions that differ only in the potential component. To obtain the unique solution, we add to this system Eq. (5.142) with $n = 2$. Analogously to the first iteration, we represent the solution in the form

$$\boldsymbol{f}^{(2)}_{\alpha,1m_1} = \boldsymbol{f}^{(t,2)}_{\alpha,1m_1} + \boldsymbol{f}^{(r,2)}_{\alpha,1m_1} + \boldsymbol{f}^{(ext,2)}_{\alpha,1m_1} + \boldsymbol{f}^{(p,2)}_{\alpha,1m_1}, \tag{5.147}$$

where

$$\boldsymbol{f}^{(\epsilon,2)}_{\alpha,1m_1} = -\frac{2}{3}\,\xi_\alpha \left\{ 4\boldsymbol{b}^{(\epsilon,2)}_{\alpha,m} + \left(\boldsymbol{b}^{(\epsilon,2)}_{\alpha,m}\right)^T \right\}, \qquad \epsilon = t, r, ext, \tag{5.148}$$

$$\boldsymbol{f}^{(p,2)}_{\alpha,1m_1} = \boldsymbol{e}^*_m \frac{Y^{(2)}_\alpha}{3}, \tag{5.149}$$

$$\boldsymbol{b}^{(\epsilon,2)}_{\alpha,m_1} = \sum_{\beta\neq\alpha} \sum_{l_2m_2} \boldsymbol{T}^{\beta,l_2m_2}_{\alpha,1m_1} \cdot \boldsymbol{f}^{(\epsilon,1)}_{\beta,l_2m_2}, \qquad \epsilon = t, r, ext. \tag{5.150}$$

According to (5.148) and (5.150), the quantities $\boldsymbol{f}^{(t,2)}_{\alpha,1m_1}$ and $\boldsymbol{f}^{(r,2)}_{\alpha,1m_1}$ are infinite power series in the parameter σ starting from the third and forth powers, respectively.

For $l_1 \geq 2$, the solution of system (5.141) can be represented in the form (5.32) with $n = 2$, where

$$\boldsymbol{f}^{(\epsilon,2)}_{\alpha,l_1m_1} = -\sum_{\beta\neq\alpha}\sum_{m_3=-l_1}^{l_1}\sum_{l_2m_2}\tilde{\boldsymbol{T}}^{\alpha,l_1m_3}_{\alpha,l_1m_1}\cdot\boldsymbol{T}^{\beta,l_2m_2}_{\alpha,l_1m_3}\cdot\boldsymbol{f}^{(\epsilon,1)}_{\beta,l_2m_2}, \qquad \epsilon = t, r, ext. \tag{5.151}$$

The quantities $\boldsymbol{f}^{(t,2)}_{\alpha,l_1m_1}$ and $\boldsymbol{f}^{(r,2)}_{\alpha,l_1m_1}$ are infinite power series in the parameter σ starting from powers of $(2 + l_1)$ and $(3 + l_1)$, respectively.

Analogously, we can determine the harmonics of the induced surface force density (and, hence, the velocity and pressure fields of the fluid) for higher iterations.

In the investigation of analogous problems, the main question is connected with the substantiation of the reduction of the infinite system of equations in unknown quantities [27,28] (for the considered approach, the unknown harmonics $\boldsymbol{f}_{\alpha,lm}$). Within the framework of this approach, we can reformulate this question as follows: How many iterations must be carried out to obtain the velocity and pressure fields of the fluid, the forces and torques exerted by the fluid on particles immersed in it up to σ^p, where p is a certain positive integer? Using the results obtained above, we can show that for any iteration $n \geq 1$, the main terms of the quantities $\boldsymbol{f}^{(t,n)}_{\alpha,lm}$, $\boldsymbol{f}^{(ext,n)}_{\alpha,lm}$, and $\boldsymbol{f}^{(r,n)}_{\alpha,lm}$ are proportional, respectively, to σ^{l+n} and σ^{l+n+1}, moreover, starting from $n = 2$ [for $\boldsymbol{f}^{(ext,n)}_{\alpha,lm}$, starting from $n = 0$], these quantities contain infinite sums of higher powers of σ. For this reason, to obtain the induced velocity and pressure fields of the fluid up to σ^p, it is necessary:

(i) to carry out p iterations,

(ii) for each sth iteration, where $s \leq p$, to retain only harmonics with $l = 0, 1, \ldots, p - s$,

(iii) in each retained harmonic, to retain all terms up to terms proportional to σ^p inclusively.

Further simplification of the results depends on the point of observation.

Substituting (5.143) and (5.144) into (5.22) and (5.147), (5.148), and (5.150) into (5.23) and taking relation (5.97) into account, after certain transformations, we obtain the following relations for the forces and torques exerted by the fluid on particles due to their translational motion and rotation corresponding to the second iteration:

$$
\begin{aligned}
\boldsymbol{F}_\alpha^{(t,2)} &= -\xi_\alpha \sum_{\beta\neq\alpha} \Bigg\{ \xi_\beta \boldsymbol{T}_M(\boldsymbol{R}_{\alpha\beta}) \cdot \sum_{\gamma\neq\beta} \xi_\gamma \boldsymbol{T}_M(\boldsymbol{R}_{\beta\gamma}) \cdot \boldsymbol{U}_\gamma^{(t)} + \frac{2}{3}\,\xi_\beta \sum_{m_2=-1}^{1} \boldsymbol{T}_{\alpha,00}^{\beta,1m_2} \\
&\quad \cdot \left\{ 4\boldsymbol{b}_{\beta,m_2}^{(t,1)} + \left(\boldsymbol{b}_{\beta,m_2}^{(t,1)}\right)^T \right\} - \sum_{l_2=2}^{\infty} \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}_{\alpha,00}^{\beta,l_2m_2} \cdot \boldsymbol{f}_{\beta,l_2m_2}^{(t,1)} \Bigg\}, \qquad (5.152)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F}_\alpha^{(r,2)} &= \xi_\alpha \sum_{\beta\neq\alpha} \Bigg\{ \boldsymbol{T}_M(\boldsymbol{R}_{\alpha\beta}) \cdot \sum_{\gamma\neq\beta} \xi_\gamma a_\gamma \sigma_{\beta\gamma}\sigma_{\gamma\beta} \left(\boldsymbol{n}_{\beta\gamma} \times \boldsymbol{\Omega}_\gamma\right) - \frac{2}{3}\,\xi_\beta \sum_{m_2=-1}^{1} \boldsymbol{T}_{\alpha,00}^{\beta,1m_2} \\
&\quad \cdot \left\{ 4\boldsymbol{b}_{\beta,m_2}^{(r,1)} + \left(\boldsymbol{b}_{\beta,m_2}^{(r,1)}\right)^T \right\} + \sum_{l_2=2}^{\infty} \sum_{m_2=-l_2}^{l_2} \boldsymbol{T}_{\alpha,00}^{\beta,l_2m_2} \cdot \boldsymbol{f}_{\beta,l_2m_2}^{(r,1)} \Bigg\}, \qquad (5.153)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{T}_\alpha^{(t,2)} &= a_\alpha \sum_{\beta\neq\alpha} \xi_\beta \sigma_{\alpha\beta}^2 \Bigg\{ \sum_{\gamma\neq\beta} \xi_\gamma \left(\boldsymbol{n}_{\alpha\beta} \times \left(\boldsymbol{T}_M(\boldsymbol{R}_{\beta\gamma}) \cdot \boldsymbol{U}_\gamma^{(t)} \right) \right) + \frac{2}{3}\,\sigma_{\beta\alpha} \sum_{m_2=-1}^{1} \sum_{m=-2}^{2} \Bigg(\boldsymbol{W}_{00,2m}^{1m_2} \\
&\quad \times \left(4\boldsymbol{b}_{\beta,m_2}^{(t,1)} + \left(\boldsymbol{b}_{\beta,m_2}^{(t,1)}\right)^T \right) \Bigg) Y_{2m}(\Theta_{\alpha\beta}, \Phi_{\alpha\beta}) - \frac{4\pi}{\xi_\beta} \sum_{l_2=2}^{\infty} \sum_{m_2=-l_2}^{l_2} \sum_{m=-(l_2+1)}^{l_2+1} \sigma_{\beta\alpha}^{l_2} \\
&\quad \times \left(\boldsymbol{W}_{00,l_2+1,m}^{l_2m_2} \times \boldsymbol{f}_{\beta,l_2m_2}^{(t,1)} \right) Y_{l_2+1,m}(\Theta_{\alpha\beta}, \Phi_{\alpha\beta}) \Bigg\}, \qquad (5.154)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{T}_\alpha^{(r,2)} &= -a_\alpha \sum_{\beta\neq\alpha} \sigma_{\alpha\beta}^2 \Bigg\{ \xi_\alpha a_\alpha \sigma_{\alpha\beta}\sigma_{\beta\alpha} \left(\boldsymbol{I} - \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta} \right) \cdot \boldsymbol{\Omega}_\alpha - \Theta(N-3) \sum_{\gamma\neq\alpha\beta} \xi_\gamma a_\gamma \sigma_{\beta\gamma}\sigma_{\gamma\beta} \\
&\quad \times \left(\boldsymbol{n}_{\alpha\beta} \times \left(\boldsymbol{n}_{\gamma\beta} \times \boldsymbol{\Omega}_\gamma \right) \right) - \frac{2}{3}\,\xi_\beta \sigma_{\beta\alpha} \sum_{m_2=-1}^{1} \sum_{m=-2}^{2} \left(\boldsymbol{W}_{00,2m}^{1m_2} \times \left(4\boldsymbol{b}_{\beta,m_2}^{(r,1)} + \left(\boldsymbol{b}_{\beta,m_2}^{(r,1)}\right)^T \right) \right) \\
&\quad \times Y_{2m}(\Theta_{\alpha\beta}, \Phi_{\alpha\beta}) + 4\pi \sum_{l_2=2}^{\infty} \sigma_{\beta\alpha}^{l_2} \sum_{m_2=-l_2}^{l_2} \sum_{m=-(l_2+1)}^{l_2+1} \left(\boldsymbol{W}_{00,l_2+1,m}^{l_2m_2} \times \boldsymbol{f}_{\beta,l_2m_2}^{(r,1)} \right) \\
&\quad \times Y_{l_2+1,m}(\Theta_{\alpha\beta}, \Phi_{\alpha\beta}) \Bigg\}. \qquad (5.155)
\end{aligned}
$$

If we restrict our consideration to the second iteration, then only the main terms in powers of σ should be taken into account in relations (5.152)–(5.154). As a result, we get

$$
\begin{aligned}
\boldsymbol{F}_\alpha^{(t,2)} &\approx -\xi_\alpha \frac{9}{16} \sum_{\gamma\neq\alpha} \sigma_{\gamma\alpha} \Bigg\{ \sigma_{\alpha\gamma} \left(\boldsymbol{I} + 3\boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma} \right) \cdot \boldsymbol{U}_\alpha^{(t)} \\
&\qquad + \Theta(N-3) \left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma} \right) \cdot \sum_{\beta\neq\gamma,\alpha} \sigma_{\beta\gamma} \left(\boldsymbol{I} + \boldsymbol{n}_{\beta\gamma}\boldsymbol{n}_{\beta\gamma} \right) \cdot \boldsymbol{U}_\beta^{(t)} \Bigg\}, \qquad (5.156)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F}_\alpha^{(r,2)} &\approx -\frac{3}{4} \sum_{\gamma\neq\alpha} \sigma_{\alpha\gamma} \Bigg\{ \xi_\alpha a_\alpha \sigma_{\alpha\gamma}\sigma_{\gamma\alpha} \left(\boldsymbol{n}_{\alpha\gamma} \times \boldsymbol{\Omega}_\alpha \right) \\
&\qquad + \Theta(N-3) \left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma} \right) \cdot \sum_{\beta\neq\gamma,\alpha} \xi_\beta a_\beta \sigma_{\beta\gamma}\sigma_{\gamma\beta} \left(\boldsymbol{n}_{\beta\gamma} \times \boldsymbol{\Omega}_\beta \right) \Bigg\}, \qquad (5.157)
\end{aligned}
$$

$$
\boldsymbol{T}_\alpha^{(t,2)} \approx \xi_\alpha a_\alpha \frac{3}{4} \sum_{\gamma\neq\alpha} \sigma_{\alpha\gamma}\sigma_{\gamma\alpha} \Bigg\{ \sigma_{\alpha\gamma} \left(\boldsymbol{n}_{\alpha\gamma} \times \boldsymbol{U}_\alpha^{(t)} \right)
$$

$$
+\,\Theta(N-3)\sum_{\beta\neq\gamma,\alpha}\sigma_{\beta\gamma}\Big(\boldsymbol{n}_{\alpha\gamma}\times\Big((\boldsymbol{I}+\boldsymbol{n}_{\beta\gamma}\boldsymbol{n}_{\beta\gamma})\cdot\boldsymbol{U}_{\beta}^{(t)}\Big)\Big)\Big\}, \tag{5.158}
$$

$$
\boldsymbol{T}_{\alpha}^{(r,2)}\approx -a_{\alpha}\sum_{\gamma\neq\alpha}\sigma_{\alpha\gamma}^{2}\Big\{\xi_{\alpha}a_{\alpha}\sigma_{\alpha\gamma}\sigma_{\gamma\alpha}\left(\boldsymbol{I}-\boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma}\right)\cdot\boldsymbol{\Omega}_{\alpha}
$$

$$
-\,\Theta(N-3)\sum_{\beta\neq\gamma,\alpha}\xi_{\beta}a_{\beta}\sigma_{\beta\gamma}\sigma_{\gamma\beta}\Big(\boldsymbol{n}_{\alpha\gamma}\times(\boldsymbol{n}_{\beta\gamma}\times\boldsymbol{\Omega}_{\beta})\Big)\Big\}. \tag{5.159}
$$

The first terms in (5.156)–(5.159) correspond to the self-interaction of particle α due to the action of the fluid induced by this particle and reflected from the rest particles. The second terms in these relations, which are nonequal to zero only for $N \geq 3$, correspond to three-particle interaction. For fixed α and β, these terms describe the contribution to the force and torque acting on particle α by the fluid induced by particle β and scattered by all rest particles (except for α). For a system of two spheres, relations (5.156)–(5.159) agree with the well-known results given in [1,12]

Taking relations (5.132)–(5.135) and (5.156)–(5.159) into account, we can represent the velocity of translational motion $\tilde{\boldsymbol{U}}_{\alpha}$ and angular velocity $\tilde{\boldsymbol{\Omega}}_{\alpha}$ of particle α in a fluid in given force fields in terms of the velocities $\boldsymbol{U}_{\beta}$ and the angular velocities $\boldsymbol{\Omega}_{\beta}$ of noninteracting particles immersed in the fluid in these fields as follows:

$$
\tilde{\boldsymbol{U}}_{\alpha}=\boldsymbol{U}_{\alpha}+\sum_{\beta=1}^{N}\boldsymbol{U}_{\alpha\beta}, \tag{5.160}
$$

$$
\tilde{\boldsymbol{\Omega}}_{\alpha}=\boldsymbol{\Omega}_{\alpha}+\sum_{\beta=1}^{N}\boldsymbol{\Omega}_{\alpha\beta}. \tag{5.161}
$$

For $\beta \neq \alpha$

$$
\boldsymbol{U}_{\alpha\beta}=\boldsymbol{U}_{\alpha\beta}^{(1)}-\Theta(N-3)\,\frac{3}{4}\sum_{\gamma\neq\alpha,\beta}\sigma_{\beta\gamma}\Big\{\frac{3}{4}\sigma_{\gamma\alpha}\left(\boldsymbol{I}+\boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma}\right)\cdot\left(\boldsymbol{I}+\boldsymbol{n}_{\beta\gamma}\boldsymbol{n}_{\beta\gamma}\right)\cdot\boldsymbol{U}_{\beta}^{(t)}
$$

$$
+\,a_{\beta}\sigma_{\beta\gamma}\sigma_{\gamma\beta}\left(\boldsymbol{I}+\boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma}\right)\cdot\left(\boldsymbol{n}_{\beta\gamma}\times\boldsymbol{\Omega}_{\beta}\right)\Big\}, \tag{5.162}
$$

$$
\boldsymbol{\Omega}_{\alpha\beta}=\boldsymbol{\Omega}_{\alpha\beta}^{(1)}+\Theta(N-3)\,\frac{3}{4a_{\alpha}}\sum_{\gamma\neq\alpha,\beta}\sigma_{\alpha\gamma}\sigma_{\beta\gamma}\sigma_{\gamma\alpha}\Big\{a_{\beta}\sigma_{\beta\gamma}\Big(\boldsymbol{n}_{\alpha\gamma}\times(\boldsymbol{n}_{\beta\gamma}\times\boldsymbol{\Omega}_{\beta})\Big)
$$

$$
+\,\frac{3}{4}\Big\{\Big(\boldsymbol{n}_{\alpha\gamma}\times\boldsymbol{U}_{\beta}^{(t)}\Big)+\Big(\boldsymbol{n}_{\beta\gamma}\cdot\boldsymbol{U}_{\beta}^{(t)}\Big)\left(\boldsymbol{n}_{\alpha\gamma}\times\boldsymbol{n}_{\beta\gamma}\right)\Big\}\Big\} \tag{5.163}
$$

are, respectively, the velocity of the translational motion of particle α and its angular velocity induced due to motion and rotation of particle β. The first terms in (5.162) and (5.163) characterizing two-particle interaction are determined by relations (5.139) and (5.140), respectively, describing the corresponding changes in the velocities of particles within the framework of the first iteration. The second terms in relations (5.162) and (5.163) characterize the changes in the translational velocity of particles and their angular velocities due to three-particle interaction. The power of the parameter σ contained in these terms is greater than that in the first terms by one. The second iteration not only changes the quantities $\boldsymbol{U}_{\alpha\beta}$ and $\boldsymbol{\Omega}_{\alpha\beta}$ but also leads to the appearance of new terms $\boldsymbol{U}_{\alpha\alpha}$ and $\boldsymbol{\Omega}_{\alpha\alpha}$ equal to

$$\boldsymbol{U}_{\alpha\alpha} = -\frac{3}{4}\sum_{\gamma\neq\alpha}\sigma_{\alpha\gamma}\sigma_{\gamma\alpha}\left\{\frac{3}{4}\left(\boldsymbol{I}+3\boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma}\right)\cdot\boldsymbol{U}_{\alpha}^{(t)} + a_{\alpha}\sigma_{\alpha\gamma}\left(\boldsymbol{n}_{\alpha\gamma}\times\boldsymbol{\Omega}_{\alpha}\right)\right\}, \tag{5.164}$$

$$\boldsymbol{\Omega}_{\alpha\alpha} = \frac{3}{4a_{\alpha}}\sum_{\gamma\neq\alpha}\sigma_{\alpha\gamma}^{2}\sigma_{\gamma\alpha}\left\{a_{\alpha}\sigma_{\alpha\gamma}\Big(\boldsymbol{n}_{\alpha\gamma}\times\left(\boldsymbol{n}_{\alpha\gamma}\times\boldsymbol{\Omega}_{\alpha}\right)\Big) + \frac{3}{4}\left(\boldsymbol{n}_{\alpha\gamma}\times\boldsymbol{U}_{\alpha}^{(t)}\right)\right\}, \tag{5.165}$$

which are absent for the first iteration and characterize the changes in the velocity of translational motion $\boldsymbol{U}_{\alpha}$ and the angular velocity $\boldsymbol{\Omega}_{\alpha}$ of particle α due to its motion in the presence of other (not necessary moving or rotating) particles. The second term in (5.165) agrees with the expression given in [1] for the angular velocity of a sphere that can freely rotate in the fluid due to its motion with constant velocity $\boldsymbol{U}_{\alpha}$ in the presence of other spheres. Note that relations (5.164) and (5.165) also follows from the terms in relations (5.162) and (5.163) proportional to $\Theta(N-3)$ by formally replacing $\Theta(N-3)$ by 1 and putting $\beta=\alpha$.

D. Friction and Mobility Tensors

Within the framework of the second iteration, using relations (5.22), (5.23), (5.92), (5.93), (5.98), (5.99), (5.132)–(5.135), and (5.156)–(5.159), we represent the forces $\boldsymbol{F}_{\alpha}^{(t)}$ and $\boldsymbol{F}_{\alpha}^{(r)}$ and the torques $\boldsymbol{T}_{\alpha}^{(t)}$ and $\boldsymbol{T}_{\alpha}^{(r)}$ exerted by the fluid on particle α as follows:

$$\boldsymbol{F}_{\alpha}^{(t)} = \sum_{k=0}^{2}\boldsymbol{F}_{\alpha}^{(t,k)} = -\sum_{\beta=1}^{N}\boldsymbol{\xi}_{\alpha\beta}^{TT}\cdot\boldsymbol{U}_{\beta}^{(t)}, \tag{5.166}$$

$$\boldsymbol{F}_{\alpha}^{(r)} = \sum_{k=0}^{2}\boldsymbol{F}_{\alpha}^{(r,k)} = -\sum_{\beta=1}^{N}\boldsymbol{\xi}_{\alpha\beta}^{TR}\cdot\boldsymbol{\Omega}_{\beta}, \tag{5.167}$$

$$\boldsymbol{T}_{\alpha}^{(t)} = \sum_{k=0}^{2}\boldsymbol{T}_{\alpha}^{(t,k)} = -\sum_{\beta=1}^{N}\boldsymbol{\xi}_{\alpha\beta}^{RT}\cdot\boldsymbol{U}_{\beta}^{(t)}, \tag{5.168}$$

$$\boldsymbol{T}_\alpha^{(r)} = \sum_{k=0}^{2} \boldsymbol{T}_\alpha^{(r,k)} = -\sum_{\beta=1}^{N} \boldsymbol{\xi}_{\alpha\beta}^{RR} \cdot \boldsymbol{\Omega}_\beta, \tag{5.169}$$

where $\boldsymbol{\xi}_{\alpha\beta}^{TT}$ and $\boldsymbol{\xi}_{\alpha\beta}^{RR}$ are, respectively, the translational and rotational friction tensors and $\boldsymbol{\xi}_{\alpha\beta}^{TR}$ and $\boldsymbol{\xi}_{\alpha\beta}^{RT}$ are the friction tensors that couple translational motion of particles and their rotation. These quantities are infinite power series in the dimensionless parameter σ. Up to the main terms corresponding to the second iteration, these quantities have the form

$$\boldsymbol{\xi}_{\alpha\beta}^{TT} = \xi_\alpha \left(\delta_{\alpha\beta}\boldsymbol{I} + \boldsymbol{\lambda}_{\alpha\beta}^{TT}\right), \tag{5.170}$$

$$\boldsymbol{\xi}_{\alpha\beta}^{RR} = \xi_\alpha^R \left(\delta_{\alpha\beta}\boldsymbol{I} + \boldsymbol{\lambda}_{\alpha\beta}^{RR}\right), \tag{5.171}$$

$$\boldsymbol{\xi}_{\alpha\alpha}^{TR} = -\xi_\alpha a_\alpha \frac{3}{4} \sum_{\gamma\neq\alpha} \sigma_{\alpha\gamma}^2 \sigma_{\gamma\alpha} \left(\boldsymbol{e}\cdot\boldsymbol{n}_{\alpha\gamma}\right), \tag{5.172}$$

$$\boldsymbol{\xi}_{\alpha\alpha}^{RT} = -\boldsymbol{\xi}_{\alpha\alpha}^{TR} = \left(\boldsymbol{\xi}_{\alpha\alpha}^{TR}\right)^T, \tag{5.173}$$

$$\begin{aligned}\boldsymbol{\xi}_{\alpha\beta}^{TR} &= -\xi_\beta a_\beta \Big\{ \sigma_{\alpha\beta}\sigma_{\beta\alpha}\left(\boldsymbol{e}\cdot\boldsymbol{n}_{\alpha\beta}\right) \\ &\quad + \Theta(N-3)\,\frac{3}{4}\sum_{\gamma\neq\alpha,\beta} \sigma_{\alpha\gamma}\sigma_{\gamma\beta}\sigma_{\beta\gamma} \Big\{ \left(\boldsymbol{e}\cdot\boldsymbol{n}_{\beta\gamma}\right) - \boldsymbol{n}_{\alpha\gamma}\left(\boldsymbol{n}_{\alpha\gamma}\times\boldsymbol{n}_{\beta\gamma}\right)\Big\}\Big\}, \qquad \beta\neq\alpha,\end{aligned} \tag{5.174}$$

$$\begin{aligned}\boldsymbol{\xi}_{\alpha\beta}^{RT} &= -a_\alpha \xi_\alpha \Big\{ \sigma_{\alpha\beta}\sigma_{\beta\alpha}\left(\boldsymbol{e}\cdot\boldsymbol{n}_{\alpha\beta}\right) \\ &\quad - \Theta(N-3)\,\frac{3}{4}\sum_{\gamma\neq\alpha,\beta} \sigma_{\alpha\gamma}\sigma_{\gamma\alpha}\sigma_{\beta\gamma} \Big\{ \left(\boldsymbol{e}\cdot\boldsymbol{n}_{\alpha\gamma}\right) + \left(\boldsymbol{n}_{\beta\gamma}\times\boldsymbol{n}_{\alpha\gamma}\right)\boldsymbol{n}_{\beta\gamma}\Big\}\Big\}, \qquad \beta\neq\alpha,\end{aligned} \tag{5.175}$$

where $\boldsymbol{e}$ is the absolutely antisymmetric unit tensor of the third rank, $(\boldsymbol{e}\cdot\boldsymbol{n}) = e_{ijk}n_k$,

$$\boldsymbol{\lambda}_{\alpha\alpha}^{TT} = \frac{9}{16}\sum_{\gamma\neq\alpha} \sigma_{\alpha\gamma}\sigma_{\gamma\alpha}\left(\boldsymbol{I} + 3\boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma}\right), \tag{5.176}$$

$$\boldsymbol{\lambda}_{\alpha\alpha}^{RR} = \frac{3}{4}\sum_{\gamma\neq\alpha} \sigma_{\alpha\gamma}^3\sigma_{\gamma\alpha}\left(\boldsymbol{I} - \boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma}\right), \tag{5.177}$$

$$\begin{aligned}\boldsymbol{\lambda}_{\alpha\beta}^{TT} &= -\frac{3}{4}\Big\{ \sigma_{\beta\alpha}\left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right) \\ &\quad - \Theta(N-3)\,\frac{3}{4}\sum_{\gamma\neq\alpha,\beta} \sigma_{\gamma\alpha}\sigma_{\beta\gamma}\left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\gamma}\boldsymbol{n}_{\alpha\gamma}\right)\cdot\left(\boldsymbol{I} + \boldsymbol{n}_{\beta\gamma}\boldsymbol{n}_{\beta\gamma}\right)\Big\},\end{aligned} \tag{5.178}$$

$$\begin{aligned}\boldsymbol{\lambda}_{\alpha\beta}^{RR} &= \frac{1}{2}\Big\{ \sigma_{\beta\alpha}^3\left(\boldsymbol{I} - 3\boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right) \\ &\quad + \Theta(N-3)\,\frac{3}{2}\sum_{\gamma\neq\alpha,\beta} \left(\frac{a_\beta}{a_\alpha}\right)^2 \sigma_{\beta\gamma}\sigma_{\gamma\beta}\sigma_{\alpha\gamma}^2 \Big\{ \left(\boldsymbol{n}_{\alpha\gamma}\cdot\boldsymbol{n}_{\beta\gamma}\right)\boldsymbol{I} - \boldsymbol{n}_{\beta\gamma}\boldsymbol{n}_{\alpha\gamma}\Big\}\Big\}.\end{aligned} \tag{5.179}$$

Taking the explicit form of relations (5.170)–(5.179) into account, we can show that the friction tensors satisfy the Onsager symmetry relations [1]

$$\left(\boldsymbol{\xi}_{\beta\alpha}^{TT}\right)^T = \boldsymbol{\xi}_{\alpha\beta}^{TT}, \quad \left(\boldsymbol{\xi}_{\beta\alpha}^{RR}\right)^T = \boldsymbol{\xi}_{\alpha\beta}^{RR}, \quad \left(\boldsymbol{\xi}_{\beta\alpha}^{RT}\right)^T = \boldsymbol{\xi}_{\alpha\beta}^{TR}, \quad \alpha, \beta = 1, 2, \ldots, N, \tag{5.180}$$

where $\left(\boldsymbol{\xi}_{\beta\alpha}^{TR}\right)^T$ means the transposition of the matrix $\boldsymbol{\xi}_{\beta\alpha}^{TR}$ with respect to the space variables $i, j = x, y, z$.

Relations for the translational friction tensors $\boldsymbol{\xi}_{\alpha\beta}^{TT}$, where $\beta = 1, 2, \ldots, N$, coincide with the corresponding well-known relations given in [20] retaining in them terms up to the second order in σ inclusively.

According to (5.170)–(5.179), the friction tensors of translational ($\boldsymbol{\xi}_{\alpha\beta}^{TT}$) and rotational ($\boldsymbol{\xi}_{\alpha\beta}^{RR}$) motions of particles are determined up to σ^2 and σ^4, respectively, and the tensors $\boldsymbol{\xi}_{\alpha\beta}^{TR}$ and $\boldsymbol{\xi}_{\alpha\beta}^{RT}$ are determined up to σ^3. In the particular case of two particles, the friction tensors defined by (5.170)–(5.175) with $N = 2$ calculated up to the above-mentioned orders in σ agree with the results given in [12].

In view of (5.166)–(5.169), the force and the torque exerted by the fluid on particle α due to the translational motion and rotation of all particles are defined as follows:

$$\boldsymbol{\mathcal{F}}_\alpha = \boldsymbol{F}_\alpha^{(t)} + \boldsymbol{F}_\alpha^{(r)} = -\sum_{\beta=1}^{N} \left\{ \boldsymbol{\xi}_{\alpha\beta}^{TT} \cdot \boldsymbol{U}_\beta^{(t)} + \boldsymbol{\xi}_{\alpha\beta}^{TR} \cdot \boldsymbol{\Omega}_\beta \right\}, \tag{5.181}$$

$$\boldsymbol{\mathcal{T}}_\alpha = \boldsymbol{T}_\alpha^{(t)} + \boldsymbol{T}_\alpha^{(r)} = -\sum_{\beta=1}^{N} \left\{ \boldsymbol{\xi}_{\alpha\beta}^{RT} \cdot \boldsymbol{U}_\beta^{(t)} + \boldsymbol{\xi}_{\alpha\beta}^{RR} \cdot \boldsymbol{\Omega}_\beta \right\}. \tag{5.182}$$

Solving the system of equations (5.181), (5.182) for the quantities $\boldsymbol{U}_\beta^{(t)}$ and $\boldsymbol{\Omega}_\beta$, we obtain

$$\boldsymbol{U}_\alpha^{(t)} = = -\sum_{\beta=1}^{N} \left\{ \boldsymbol{\mu}_{\alpha\beta}^{TT} \cdot \boldsymbol{\mathcal{F}}_\beta + \boldsymbol{\mu}_{\alpha\beta}^{TR} \cdot \boldsymbol{\mathcal{T}}_\beta \right\}, \tag{5.183}$$

$$\boldsymbol{\Omega}_\alpha = = -\sum_{\beta=1}^{N} \left\{ \boldsymbol{\mu}_{\alpha\beta}^{RT} \cdot \boldsymbol{\mathcal{F}}_\beta + \boldsymbol{\mu}_{\alpha\beta}^{RR} \cdot \boldsymbol{\mathcal{T}}_\beta \right\}, \tag{5.184}$$

where the translational ($\boldsymbol{\mu}_{\alpha\beta}^{TT}$) and rotational ($\boldsymbol{\mu}_{\alpha\beta}^{RR}$) mobility tensors as well as the tensors $\boldsymbol{\mu}_{\alpha\beta}^{TR}$ and $\boldsymbol{\mu}_{\alpha\beta}^{RT}$ that couple translational and rotational motions of particles determined up to the same orders in the parameter σ that the corresponding friction tensors have the form

$$\boldsymbol{\mu}^{TT}_{\alpha\beta} = \frac{1}{\xi_\alpha}\left\{\delta_{\alpha\beta}\boldsymbol{I} + (1-\delta_{\alpha\beta})\,\frac{3}{4}\sigma_{\alpha\beta}\left(\boldsymbol{I} + \boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta}\right)\right\}, \tag{5.185}$$

$$\boldsymbol{\mu}^{RR}_{\alpha\beta} = \frac{1}{\xi^R_\alpha}\left\{\delta_{\alpha\beta}\boldsymbol{I} + (1-\delta_{\alpha\beta})\,\frac{\sigma^3_{\alpha\beta}}{2}\left(3\boldsymbol{n}_{\alpha\beta}\boldsymbol{n}_{\alpha\beta} - \boldsymbol{I}\right)\right\}, \tag{5.186}$$

$$\boldsymbol{\mu}^{TR}_{\alpha\beta} = (1-\delta_{\alpha\beta})\,\frac{1}{8\pi\eta R^2_{\alpha\beta}}\left(\boldsymbol{e}\cdot\boldsymbol{n}_{\alpha\beta}\right), \tag{5.187}$$

$$\boldsymbol{\mu}^{RT}_{\alpha\beta} = \boldsymbol{\mu}^{TR}_{\alpha\beta}. \tag{5.188}$$

It is easy to see that the mobility tensors defined by relations (5.185)–(5.188) satisfy the Onsager symmetry relations [1]

$$\left(\boldsymbol{\mu}^{TT}_{\beta\alpha}\right)^T = \boldsymbol{\mu}^{TT}_{\alpha\beta}, \quad \left(\boldsymbol{\mu}^{RR}_{\beta\alpha}\right)^T = \boldsymbol{\mu}^{RR}_{\alpha\beta}, \quad \left(\boldsymbol{\mu}^{RT}_{\beta\alpha}\right)^T = \boldsymbol{\mu}^{TR}_{\alpha\beta}, \quad \alpha,\beta = 1,2,\ldots,N. \tag{5.189}$$

Within the framework of the considered approximation, for which $\boldsymbol{\mu}^{TT}_{\alpha\beta}$, $\boldsymbol{\mu}^{RR}_{\alpha\beta}$, $\boldsymbol{\mu}^{RT}_{\alpha\beta}$, and $\boldsymbol{\mu}^{TR}_{\alpha\beta}$ are determined up to terms of σ^2, σ^4, σ^3, and σ^3, respectively, relations (5.185)–(5.188) for the mobility tensors coincide with known results given in [19]. Unlike the friction tensors calculated in this approximation, the mobility tensors are determined only by two-particle interactions and $\boldsymbol{\mu}^{TR}_{\alpha\alpha} = \boldsymbol{\mu}^{RT}_{\alpha\alpha} = 0$. In order to take into account three-particle interactions in the mobility tensors, it is necessary to carry out calculations for higher iterations.

VI. CONCLUSIONS

In the present paper, we have proposed the procedure for the determination of the time-dependent velocity and pressure fields of an unbounded incompressible viscous fluid in an external force field induced by an arbitrary number of spheres moving and rotating in it as well as the forces and torques exerted by the fluid on the particles. The corresponding quantities are expressed in terms of the induced surface force densities. We showed that the relations for the required harmonics of the induced surface force densities given in [30] for the stationary case are expressed in terms on nonexistent inverse tensors. We analyzed in detail the reasons for zero of the determinant of the corresponding system of equations. We formulated the consistent system of algebraic equations in the harmonics of these induced surface force densities. In the stationary case, we obtained relations for the harmonics up to the second approximation inclusively. The obtained general results for the fluid velocity and pressure and the friction and mobility tensors corresponding to the second approximation agree with the well-known results obtained by other methods in various particular cases.

The proposed procedure can be used for the investigation of hydrodynamic interactions of particles in the fluid for higher-order approximations in a similar way as it is realized in the present paper. At the same time, the relations for the fluid velocity and pressure and the forces and torques exerted by the fluid on the particles expressed in the present paper in terms of induced surface forces and the proposed procedure for the determination of these forces can be regarded as a basis for the study of hydrodynamic interactions between particles in the nonstationary case. Certain results concerning the time-dependent hydrodynamic interactions of particles in a nonstationary fluid will be given in subsequent papers.

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APPENDIX

To determine the explicit form of the quantities $F_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$, $C_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta})$, and $P_{l_1 l_2, l}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega)$ defined by relations (3.36), (3.40), and (3.42), it is necessary to take integrals of the product of three spherical Bessel functions. For this purpose, first, we consider the integral

$$\int_0^\infty dx \, \frac{x^{\alpha-1}}{x^2+\kappa^2} J_\nu(ax) J_\mu(bx) J_\gamma(cx), \tag{A1}$$

where α is real, ν, μ, and γ are real quantities nonequal to negative integers, $\mathrm{Re}\,\kappa > 0$, and a, b, and c are real positive quantities such that $c \geq a + b$. The case where the order of a Bessel function is a negative integer is reduced to the above-considered integral using the known relation $J_{-n}(x) = (-1)^n J_n(x)$ valid for integer n. To take this integral, we use the Hankel method [35]. To this end, we consider the following auxiliary integral:

$$\int_0^\infty dx \, \frac{x^{\alpha-1}}{x^2+\kappa^2} J_\nu(ax) J_\mu(bx) \left[H_\gamma^{(1)}(cx) + (-1)^p H_\gamma^{(1)}(-cx) \right], \tag{A2}$$

where $H_\gamma^{(1)}(x)$ is the Bessel function of the third kind and p is a certain quantity. For $p = 1 - \alpha - \nu - \mu$, integral (A2) can be reduced to the form

$$\int_{-\infty}^\infty dx \, \frac{x^{\alpha-1}}{x^2+\kappa^2} J_\nu(ax) J_\mu(bx) H_\gamma^{(1)}(cx). \tag{A3}$$

We consider the integral

$$\int_C dz \, \frac{z^{\alpha-1}}{z^2+\kappa^2} J_\nu(az) J_\mu(bz) H_\gamma^{(1)}(cz). \tag{A4}$$

in the plane of the complex variable z along the closed contour C consisting of the large C_R and small C_r semicircles of radii R and r, respectively, centered at the origin of coordinates and lying above the real axis, and the segments $[-R, -r]$ and $[r, R]$ along the real axis as $R \to \infty$ and $r \to 0$. Taking into account that the functions $J_\nu(z)$ and $H_\nu^{(1)}(z)$ are analytic functions of z in the entire complex plane of z with the cut $(-\infty, 0]$, we obtain that integral (A4) is determined by the simple pole of the integrand at the point $z = i\kappa$ provided that the integrals along the contours C_R and C_r are finite. Since $R \to \infty$, for $c > a + b$, by the Jordan lemma, the integral along the contour C_R is equal to zero if $\alpha < 4\frac{1}{2}$. In the case where $c = a + b$, the integral along the contour C_R is equal to zero for $\alpha < 3\frac{1}{2}$ and finite for $\alpha = 3\frac{1}{2}$. The condition of finiteness of the integral along the contour C_r as $r \to 0$ imposes the following constraint on the quantity α for $\gamma \neq 0$:

$$\alpha \geq \gamma - \nu - \mu. \tag{A5}$$

Furthermore, the integral along the infinitely small semicircle C_r is nonzero only in the case of equality (A5). In the particular case $\gamma = 0$, inequality (A5) is strict.

Passing in (A4) to the limit as $R \to \infty$ and $r \to 0$, we can represent integral (A2) in the form

$$
\int_0^\infty dx \, \frac{x^{\alpha-1}}{x^2+\kappa^2} \, J_\nu(ax) J_\mu(bx) \left[H^{(1)}_\gamma(cx) + (-1)^p H^{(1)}_\gamma(-cx) \right] = 2(i\kappa)^{\alpha-2} \, i^{\nu+\mu-\gamma} I_\nu(a\kappa) I_\mu(b\kappa) K_\gamma(c\kappa)
$$

$$
+ \, \delta_{\alpha,\gamma-\nu-\mu} \, (1-\delta_{\gamma,0}) \frac{2^\alpha \Gamma(\gamma)}{\Gamma(\nu+1)\Gamma(\mu+1)} \frac{a^\nu b^\mu}{\kappa^2 c^\gamma} - \delta_{c,a+b} \, \delta_{\alpha,\frac{7}{2}} \, \frac{i^{1+\nu+\mu-\gamma}}{2\sqrt{\pi abc}} (1+i), \qquad \text{(A6)}
$$

$$
p = 1-\alpha-\nu-\mu, \qquad \alpha \geq \gamma-\mu-\nu,
$$

$$
\alpha < \alpha_c \quad \text{if} \quad c > a+b \quad \text{or} \quad \alpha \leq \alpha_c \quad \text{if} \quad c = a+b,
$$

where

$$
\alpha_c = \begin{cases} 4\frac{1}{2}, & \text{if } c > a+b, \\ 3\frac{1}{2}, & \text{if } c = a+b \end{cases} \qquad \text{(A7)}
$$

and $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions of the first and second kinds, respectively.

To determine the required integral (A1), we note that in (A2)

$$
\left[H^{(1)}_\gamma(cx) + (-1)^p H^{(1)}_\gamma(-cx) \right] = \left[1 + (-1)^{\gamma-\nu-\mu-\alpha} \right] J_\nu(cx) + \left[1 + (-1)^{\gamma-\nu-\mu-\alpha+1} \right] Y_\nu(cx), \qquad \text{(A8)}
$$

where $Y_\nu(x)$ is the Bessel function of the second kind. In the particular case where $\alpha = \gamma - \nu - \mu + 2k$, where $k = 0, 1, 2, \ldots$, condition (A5) is true and we obtain the following result for the required integral (A1):

$$
\int_0^\infty dx \, \frac{x^{\alpha-1}}{x^2+\kappa^2} J_\nu(ax) J_\mu(bx) J_\gamma(cx) = (i\kappa)^{\alpha-2} i^{\nu+\mu-\gamma} I_\nu(a\kappa) I_\mu(b\kappa) K_\gamma(c\kappa) + \delta_{\alpha,\gamma-\nu-\mu} \, (1-\delta_{\gamma,0})
$$

$$
\times \, \frac{2^{\alpha-1}\Gamma(\gamma)}{\Gamma(\nu+1)\Gamma(\mu+1)} \frac{a^\nu b^\mu}{\kappa^2 c^\gamma} + \delta_{c,a+b} \, \delta_{\alpha,\frac{7}{2}} \, \frac{(-1)^k}{2\sqrt{2\pi abc}}, \qquad \text{(A9)}
$$

$$
\alpha = \gamma-\nu-\mu+2k, \quad k = 0, 1, 2, \ldots,
$$

$$
\alpha < \alpha_c \quad \text{if} \quad c > a+b \quad \text{or} \quad \alpha \leq \alpha_c \quad \text{if} \quad c = a+b.
$$

In the particular case where $\gamma = \nu$, $c > a+b$, and $\alpha = 2k - \mu < 9/2$, where $k = 1, 2, \ldots$, relation (A9) agrees with the known result given in [36, p. 232].

Setting in (A9) $\nu = l_1 + \frac{1}{2}$, $\mu = l_2 + \frac{1}{2}$, and $\gamma = l + \frac{1}{2}$, where l_1, l_2, and l are nonnegative integers, we get

$$\int_0^\infty dx\, \frac{x^\alpha}{x^2+\kappa^2}\, j_{l_1}(ax) j_{l_2}(bx) j_l(cx) = (-1)^{k-1}\kappa^{\alpha-1}\tilde{j}_{l_1}(a\kappa)\tilde{j}_{l_2}(b\kappa)\tilde{h}_l(c\kappa) + \delta_{\alpha,l-l_1-l_2}\,\pi^{\frac{3}{2}}2^{\alpha-3}$$

$$\times\, \frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)}\frac{a^{l_1}b^{l_2}}{\kappa^2 c^{l+1}} + \delta_{c,a+b}\,\delta_{\alpha,4}\,\frac{(-1)^k\pi}{8abc},$$

$$\alpha = l - l_1 - l_2 + 2k, \quad k = 0,1,2,\ldots, \tag{A10}$$

$$\alpha < \alpha_3 \quad \text{if} \quad c > a+b \quad \text{or} \quad \alpha \le \alpha_3 \quad \text{if} \quad c = a+b,$$

where $\tilde{j}_l(x) = \sqrt{\pi/(2x)}I_{l+\frac{1}{2}}(x)$ and $\tilde{h}_l(x) = \sqrt{\pi/(2x)}K_{l+\frac{1}{2}}(x)$ are the modified spherical Bessel functions of the first and third kind, respectively, [34] and

$$\alpha_3 = \begin{cases} 5, & \text{if } c > a+b, \\ 4, & \text{if } c = a+b. \end{cases} \tag{A11}$$

Passing in (A10) to the limit as $\kappa \to 0$, we get

$$\int_0^\infty dx\, x^{\alpha-2}\, j_{l_1}(ax) j_{l_2}(bx) j_l(cx) = \pi^{\frac{3}{2}}2^{l-(l_1+l_2+3)}\frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)}\left\{\delta_{k,1} - \delta_{k,0}\,\frac{1}{2}\left(\frac{a^2}{2l_1+3}\right.\right.$$

$$\left.\left. + \frac{b^2}{2l_2+3} - \frac{c^2}{2l^2-1}\right)\right\}\frac{a^{l_1}b^{l_2}}{c^{l+1}} + \delta_{c,a+b}\,\delta_{\alpha,4}\,\frac{(-1)^k\pi}{8abc}, \tag{A12}$$

$$c \ge a+b, \quad a,b > 0, \quad \alpha = l - l_1 - l_2 + 2k, \; k = 0,1,2,\ldots,$$

$$\alpha < \alpha_3 \quad \text{if} \quad c > a+b \quad \text{or} \quad \alpha \le \alpha_3 \quad \text{if} \quad c = a+b,$$

except for the special case where $\alpha, l_1, l_2, l = 0$.

For $c > a+b$, relation (A12) agrees with the result given in [36, p. 239].

Note that we can apply the method used above for the determination of integral (A1) to the integral

$$\int_0^\infty dx\, x^{\alpha-2}\, j_{l_1}(ax) j_{l_2}(bx) j_l(cx). \tag{A13}$$

In a similar way as for integral (A1), we get

$$\int_0^\infty dx\, x^{\alpha-2}\, j_{l_1}(ax) j_{l_2}(bx) j_l(cx) = \delta_{k,1}\,\pi^{\frac{3}{2}}2^{\alpha-5}\frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)}\frac{a^{l_1}b^{l_2}}{c^{l+1}} + \delta_{c,a+b}\,\delta_{\alpha,4}\,\frac{(-1)^k\pi}{8abc},$$

$$c \ge a+b, \quad a,b > 0, \quad \alpha = l - l_1 - l_2 + 2k, \quad k = 1,2,\ldots,$$

$$\alpha < \alpha_3 \quad \text{if} \quad c > a+b \quad \text{or} \quad \alpha \le \alpha_3 \quad \text{if} \quad c = a+b. \tag{A14}$$

Relation (A14) coincides with (A12) for $k \geq 1$. At the same time, we see that the direct application of this method to (A13) does not enable us to determine integral (A13) for $\alpha = l - l_1 - l_2$ ($k = 0$) because relation (A14) is true for $k > 0$.

Analogously, we can determine the integral of the product of two spherical Bessel functions. The final result is presented as follows:

$$
\begin{aligned}
\int_0^\infty dx \, \frac{x^\alpha}{x^2 + \kappa^2} \, j_{l_1}(bx) j_{l_2}(cx) &= (-1)^{k-1} \kappa^{\alpha-1} \tilde{j}_{l_2}(c\kappa) \tilde{h}_{l_1}(b\kappa) \\
&+ \delta_{\alpha, l_1 - l_2} \, \pi 2^{\alpha-2} \frac{\Gamma\left(l_1 + \frac{1}{2}\right)}{\Gamma\left(l_2 + \frac{3}{2}\right)} \frac{c^{l_2}}{\kappa^2 b^{l_1+1}} + \delta_{b,c} \, \delta_{\alpha,3} \, \frac{(-1)^k \pi}{4b^2}, \\
&b \geq c > 0, \qquad \alpha = l_1 - l_2 + 2k, \quad k = 0, 1, 2, \ldots, \\
&\alpha < \alpha_2 \quad \text{if} \quad b > c \quad \text{or} \quad \alpha \leq \alpha_2 \quad \text{if} \quad b = c,
\end{aligned}
\tag{A15}
$$

where

$$
\alpha_2 = \begin{cases} 4, & \text{if } b > c, \\ 3, & \text{if } b = c. \end{cases}
\tag{A16}
$$

For $b > c$ and $k > 0$, integral (A15) agrees with the result given in [36, p. 213].

Passing in (A15) to the limit as $\kappa \to 0$, we obtain the known Weber–Schafheitlin integral

$$
\begin{aligned}
\int_0^\infty dx \, x^{\alpha-2} \, j_{l_1}(bx) j_{l_2}(cx) &= \pi 2^{l_1 - l_2 - 2} \frac{c^{l_2}}{b^{l_1+1}} \frac{\Gamma\left(l_1 + \frac{1}{2}\right)}{\Gamma\left(l_2 + \frac{3}{2}\right)} \left\{ \delta_{k,1} + \delta_{k,0} \frac{1}{2} \left(\frac{b^2}{2l_1 - 1} - \frac{c^2}{2l_2 + 3} \right) \right\} \\
&+ \delta_{b,c} \, \delta_{\alpha,3} \, \frac{(-1)^k \pi}{4b^2}, \\
&b \geq c > 0, \qquad \alpha = l_1 - l_2 + 2k, \quad k = 0, 1, 2, \ldots, \\
&\alpha < \alpha_2 \quad \text{if} \quad b > c \quad \text{or} \quad \alpha \leq \alpha_2 \quad \text{if} \quad b = c,
\end{aligned}
\tag{A17}
$$

except for the special case $\alpha, l_1, l_2 = 0$.

Relation (A17) agrees with the results given in [36, pp. 239 and 209] obtained for $b > c$ and $b = c$, $\alpha = 3$, respectively.

Note that except for the particular case where $b = c$ and $\alpha < 3$, it is simpler to determine the integrals of the product of two spherical Bessel functions (A15) and (A17) on the basis of relations (A10) and (A12) for the integrals of the product of three spherical Bessel functions found for $a, b, c > 0$. For this purpose, it is necessary to pass to the limit $a = 0$ (or $b = 0$) in (A10) and (A12) defined for $\alpha < \alpha_3$ and to take into account that $\tilde{j}_l(0) = \delta_{l,0}$ and the fact that for $a = 0$ (or $b = 0$), the condition of the finiteness of integral (A4) along the contour C_r leads to the decrease in the maximum possible value of α by one, i.e., $\alpha_3 \to \alpha_2$. After the corresponding renaming, we obtain, respectively, (A15) and (A17) defined for $\alpha < \alpha_2$.

In the particular case $b = c$, it is convenient to represent integrals (A15) and (A17) as follows:

$$\int_0^\infty dx \, \frac{x^\alpha}{x^2+\kappa^2} \, j_{l_1}(bx) j_{l_2}(cx) = (-1)^{k-1} \kappa^{\alpha-1} \tilde{j}_{l_{max}}(b\kappa) \tilde{h}_{l_{min}}(b\kappa) + \delta_{k,0} \, \frac{\pi}{2(b\kappa)^2} \left(\frac{2}{b}\right)^{\alpha-1} \times \frac{\Gamma\left(l_{min}+\frac{1}{2}\right)}{\Gamma\left(l_{max}+\frac{3}{2}\right)} + \delta_{\alpha,3} \, \frac{(-1)^k \pi}{4b^2}, \quad \alpha = l_{min} - l_{max} + 2k, \quad k = 0, 1, 2, \ldots, \tag{A18}$$

$$\int_0^\infty dx \, x^{\alpha-2} \, j_{l_1}(bx) j_{l_2}(bx) = \frac{\pi}{4b} \left(\frac{b}{2}\right)^{2k-\alpha} \frac{\Gamma\left(l_{min}+\frac{1}{2}\right)}{\Gamma\left(l_{max}+\frac{3}{2}\right)} \left\{ \delta_{k,1} + \delta_{k,0} \left(\frac{1}{2l_{min}-1} - \frac{1}{2l_{max}+3} \right) \frac{b^2}{2} \right\} + \delta_{\alpha,3} \, \frac{(-1)^k \pi}{4b^2}, \qquad \alpha = l_{min} - l_{max} + 2k < 3, \quad k = 0, 1, 2, \ldots, \tag{A19}$$

where $l_{max} = \max(l_1, l_2)$ and $l_{min} = \min(l_1, l_2)$.

To determine the quantity $F_{l_1 l_1, l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$ defined by relation (3.36) for $l = l_1 + l_2 - 2p \geq 0$, where $p = -1, 0, 1, \ldots, p_{max}$, we use the above obtained integrals (A10) (for $r_\beta \neq 0$) and (A15) (for $r_\beta = 0$) setting $\alpha = 2$ in them. Indeed, in this case, the representation of α in the form $\alpha = l - l_1 - l_2 + 2k$, where $k = 0, 1, 2, \ldots$, corresponds to $k = 1 + p$. Since $0 \geq r_\beta \geq a_\beta$, depending on the distance r_α to the point of observation, two cases are possible, namely, $r_\alpha \leq R_{\alpha\beta} - r_\beta$ and $r_\alpha \geq R_{\alpha\beta} + r_\beta$ [the domain $R_{\alpha\beta} + r_\beta > r_\alpha > R_{\alpha\beta} - r_\beta$ should be considered separately because it does not satisfy the condition $c \geq a + b$ of integral (A10)]. Depending on the case, $R_{\alpha\beta}$ or r_α should be taken as the quantity c in relation (A10). As a result, we obtain

$$F_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega) = (-1)^p \frac{2\kappa}{\pi\eta} \left\{ \tilde{j}_{l_1}(x_\alpha) \tilde{j}_{l_2}(x_\beta) \tilde{h}_l(y_{\alpha\beta}) - \delta_{l, l_1+l_2+2} \, \frac{\pi^{3/2}}{2} \frac{\Gamma\left(l_1+l_2+\frac{5}{2}\right)}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)} \times \frac{r_\alpha^{l_1} r_\beta^{l_2}}{y_{\alpha\beta}^3 R_{\alpha\beta}^{l_1+l_2}} \right\}, \quad l = l_1 + l_2 - 2p \geq 0, \quad p = -1, 0, 1, \ldots, p_{max}, \tag{A20}$$

for $r_\alpha \le R_{\alpha\beta} - r_\beta$ and

$$F_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega) = (-1)^{l_2-p}\frac{2\kappa}{\pi\eta}\left\{\tilde{h}_{l_1}(x_\alpha)\tilde{j}_{l_2}(x_\beta)\tilde{j}_l(y_{\alpha\beta}) - \delta_{l,l_1-l_2-2}\frac{\pi^{3/2}}{2}\right.$$
$$\left.\times\frac{\Gamma\left(l_1+\frac{1}{2}\right)}{\Gamma\left(l_2+\frac{3}{2}\right)\Gamma\left(l_1-l_2-\frac{1}{2}\right)}\frac{r_\beta^{l_2}R_{\alpha\beta}^{l}}{x_\alpha^3 r_\alpha^{l_1-2}}\right\},$$
$$l = l_1 + l_2 - 2p \ge 0, \quad p = -1, 0, 1, \dots, p_{max}, \tag{A21}$$

for $r_\alpha \ge R_{\alpha\beta} + r_\beta$. Here, $x_\alpha = \kappa r_\alpha$, $x_\beta = \kappa r_\beta$, and $y_{\alpha\beta} = \kappa R_{\alpha\beta}$.

Note that in view of the representation (A10) for the integral of the product of three spherical Bessel functions, relation (A21) for the quantity $F_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$ valid for $r_\alpha \ge R_{\alpha\beta} + r_\beta$ can also be represented in the form (A20) valid for $r_\alpha \le R_{\alpha\beta} - r_\beta$ with the changes $r_\alpha \leftrightarrow R_{\alpha\beta}$ and $l_1 \leftrightarrow l$ putting $l_1 = l + l_2 - 2s$ (i.e., $p = l_2 - s$), where $s = -1, 0, 1, \dots, s_{max}$ and $s_{max} = \min\Big(\left[(l+l_2)/2\right], 1 + \min(l, l_2)\Big)$.

Putting $r_\beta = 0$ and $r_\alpha = R_{\alpha\beta}$ in (A20) and (A21) and equating the obtained relations to one another, we obtain the following relation for the modified spherical Bessel functions:

$$\tilde{j}_l(x)\tilde{h}_{l+2}(x) - \tilde{j}_{l+2}(x)\tilde{h}_l(x) = \frac{\pi}{x^3}\left(l + \frac{3}{2}\right), \qquad l \ge 0. \tag{A22}$$

Note that this relation can also be derived by equating relations (A.15) for $c = b$ and (A.18) putting $\alpha = 2$ and $l_1 = l_2 + 2$ in them.

To determine the quantity $C_{l_1 l_1,l}(r_\alpha, r_\beta, R_{\alpha\beta})$ defined by relation (3.40) for $l = l_1 + l_2 - 2p + 1 \ge 0$, where $p = 0, 1, \dots, \tilde{p}_{max}$, we use relation (A12) with $\alpha = 3$ and take into account that $r_\beta \le a_\beta$. As a result, for $r_\beta \ne 0$, we get

$$C_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}) = \delta_{l,l_1+l_2+1}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(l_1+l_2+\frac{3}{2}\right)}{\Gamma\left(l_1+\frac{3}{2}\right)\Gamma\left(l_2+\frac{3}{2}\right)}\frac{r_\alpha^{l_1}r_\beta^{l_2}}{R_{\alpha\beta}^{l_1+l_2+2}} \tag{A23}$$

for $r_\alpha \le R_{\alpha\beta} - r_\beta$ and

$$C_{l_1 l_2,l}(r_\alpha, r_\beta, R_{\alpha\beta}) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(l_1+\frac{1}{2}\right)}{\Gamma\left(l_2+\frac{3}{2}\right)}\frac{r_\beta^{l_2}R_{\alpha\beta}^{l_1-l_2-1}}{r_\alpha^{l_1+1}}\left\{\delta_{l,l_1-l_2-1}\frac{1}{\Gamma\left(l_1-l_2+\frac{1}{2}\right)} + \delta_{l,l_1-l_2-3}\right.$$
$$\left.\times\frac{2}{\Gamma\left(l_1-l_2-\frac{3}{2}\right)}\left[\frac{1}{2l_1-1}\left(\frac{r_\alpha}{R_{\alpha\beta}}\right)^2 - \frac{1}{2l_2+3}\left(\frac{r_\beta}{R_{\alpha\beta}}\right)^2 - \frac{1}{2l+3}\right]\right\} \tag{A24}$$

for $r_\alpha \geq R_{\alpha\beta} + r_\beta$.

Just as for the representation of the quantity $F_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta}, \omega)$ for $r_\alpha \geq R_{\alpha\beta} + r_\beta$, we can show the possibility of the representation of relation (A24) for the quantity $C_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta})$ in the form (A23) valid for $r_\alpha \leq R_{\alpha\beta} - r_\beta$ with the changes $r_\alpha \leftrightarrow R_{\alpha\beta}$ and $l_1 \leftrightarrow l$ putting, $l_1 = l + l_2 + 1 - 2s$, where $s = 0, 1, \ldots, \tilde{s}_{max}$ and $\tilde{s}_{max} = \min\Big([(l + l_2 + 1)/2]\,, 1 + \min(l, l_2)\Big)$.

In the particular case $r_\beta = 0$, to determine the quantity $C_{l_1 l_2, l}(r_\alpha, 0, R_{\alpha\beta})$, it is necessary to use relation (A17). As a result, we obtain

$$C_{l_1 l_2, l}(r_\alpha, 0, R_{\alpha\beta}) = \delta_{l_2, 0} C_{l_1, l}(r_\alpha, R_{\alpha\beta}), \qquad l = l_1 \pm 1 \geq 0, \tag{A25}$$

where

$$C_{l_1, l}(r_\alpha, R_{\alpha\beta}) = \frac{1}{R_{\alpha\beta}^2} \left\{ \delta_{l, l_1+1} \left(\frac{r_\alpha}{R_{\alpha\beta}} \right)^{l_1} + \delta_{r_\alpha, R_{\alpha\beta}} \frac{1}{2} \left(\delta_{l, l_1-1} - \delta_{l, l_1+1} \right) \right\} \tag{A26}$$

for $r_\alpha \leq R_{\alpha\beta}$ and

$$C_{l_1, l}(r_\alpha, R_{\alpha\beta}) = \frac{1}{r_\alpha^2} \left\{ \delta_{l, l_1-1} \left(\frac{R_{\alpha\beta}}{r_\alpha} \right)^{l_1-1} + \delta_{r_\alpha, R_{\alpha\beta}} \frac{1}{2} \left(\delta_{l, l_1+1} - \delta_{l, l_1-1} \right) \right\} \tag{A27}$$

for $r_\alpha \geq R_{\alpha\beta}$. In this case, the quantities $C_{l_1, l}(r_\alpha, R_{\alpha\beta})$ should be considered only for $l = l_1 \pm 1 \geq 0$ because for $l_2 = 0$, $\tilde{p}_{max} = 0$ if $l_1 = 0$ or $\tilde{p}_{max} = 1$ if $l_1 \geq 1$.

Note that simply passing to the limit in (A23) and (A24) as $r_\beta \to 0$, we obtain

$$C_{l_1 l_2, l}(r_\alpha, 0, R_{\alpha\beta}) = \delta_{l_2, 0} \tilde{C}_{l_1, l}(r_\alpha, R_{\alpha\beta}), \qquad l = l_1 \pm 1 \geq 0, \tag{A28}$$

where

$$\tilde{C}_{l_1, l}(r_\alpha, R_{\alpha\beta}) = \delta_{l, l_1+1} \frac{1}{R_{\alpha\beta}^2} \left(\frac{r_\alpha}{R_{\alpha\beta}} \right)^{l_1} \tag{A29}$$

for $r_\alpha \leq R_{\alpha\beta}$ and

$$\tilde{C}_{l_1, l}(r_\alpha, R_{\alpha\beta}) = \delta_{l, l_1-1} \frac{1}{r_\alpha^2} \left(\frac{R_{\alpha\beta}}{r_\alpha} \right)^{l_1-1} \tag{A30}$$

for $r_\alpha \geq R_{\alpha\beta}$.

Comparing relations (A29) and (A30) with relations (A26) and (A27), we see that the function $\tilde{C}_{l_1,l}(r_\alpha, R_{\alpha\beta})$ corresponds to the first term of the function $C_{l_1,l}(r_\alpha, R_{\alpha\beta})$. Furthermore, in view of (A29) and (A30), the function $\tilde{C}_{l_1,l}(r_\alpha, R_{\alpha\beta})$ has a discontinuity at the point $r_\alpha = R_{\alpha\beta}$, while, according to (A26) and (A27), the function $C_{l_1,l}(r_\alpha, R_{\alpha\beta})$ is continuous at this point and equal to

$$C_{l_1,l}(R_{\alpha\beta}, R_{\alpha\beta}) = \frac{1}{2R_{\alpha\beta}^2}\left(\delta_{l,l_1+1} + \delta_{l,l_1-1}\right), \qquad l = l_1 \pm 1 \geq 0. \tag{A31}$$

In a similar way, using (A10), we can find the quantity $P_{l_1 2,l}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega)$ defined by relation (3.42) with $l_2 = 2$. Indeed, according to (3.31) and (3.51), this quantity should be determined only for $l = l_1 \pm 1 \geq 0$, which agrees with the constraint for the quantities l_1, l_2, l, α, and k in the integral defined by relation (A10). As a result, we get

$$P_{l_1 2,l}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega) = (-1)^p \frac{2}{\pi\eta}\,\tilde{j}_{l_1}(x_\alpha)\tilde{j}_2(b_\beta)\tilde{h}_l(y_{\alpha\beta}) \tag{A32}$$

for $r_\alpha \leq R_{\alpha\beta} - a_\beta$ and

$$P_{l_1 2,l}(r_\alpha, a_\beta, R_{\alpha\beta}, \omega) = (-1)^{p+1} \frac{2}{\pi\eta}\,\tilde{h}_{l_1}(x_\alpha)\tilde{j}_2(b_\beta)\tilde{j}_l(y_{\alpha\beta}) \tag{A33}$$

for $r_\alpha \geq R_{\alpha\beta} + a_\beta$. Here, $b_\beta = \kappa a_\beta$.

Using the obtained integrals of the product of two spherical Bessel functions (A15) and (A17), we can determine the quantities $F_{l_1 l_2}(r_\alpha, r'_\alpha, \omega)$, $C_{l_1 l_2}(r_\alpha, r'_\alpha)$, and $P_{1,2}(r_\alpha, a_\alpha, \omega)$, defined, respectively, by relations (3.64), (3.66), and (3.67), for $r_\alpha \geq a_\alpha$ and $0 \leq r'_\alpha \leq a_\alpha$, taking into account that $F_{l_1,l_2}(r_\alpha, r'_\alpha, \omega)$ and $C_{l_1 l_2}(r_\alpha, r'_\alpha)$ should be determined only for $l_2 = l_1 + 2p \geq 0$, where $p = 0, \pm 1$, and $l_2 = l_1 \pm 1 \geq 0$, respectively. As a result, we get

$$F_{l_1 l_2}(r_\alpha, r'_\alpha, \omega) = (-1)^p \frac{2\kappa}{\pi\eta}\left\{\tilde{h}_{l_1}(x_\alpha)\tilde{j}_{l_2}(x'_\alpha) - \delta_{l_2,l_1-2}\left(l_1 - \frac{1}{2}\right)\frac{\pi}{x_\alpha^3}\left(\frac{r'_\alpha}{r_\alpha}\right)^{l_1-2}\right\},$$
$$l_2 = l_1 + 2p \geq 0, \quad p = 0, \pm 1, \tag{A34}$$

$$C_{l_1 l_2}(r_\alpha, r'_\alpha) = \delta_{l_2,l_1-1}\frac{1}{r_\alpha^2}\left(\frac{r'_\alpha}{r_\alpha}\right)^{l_1-1} + \delta_{r_\alpha,a}\,\delta_{r'_\alpha,a}\,\frac{1}{2a_\alpha^2}\left(\delta_{l_2,l_1+1} - \delta_{l_2,l_1-1}\right),$$
$$l_2 = l_1 \pm 1 \geq 0, \tag{A35}$$

$$P_{1,2}(r_\alpha, a_\alpha, \omega) = \frac{2}{\pi\eta}\,\tilde{h}_1(x_\alpha)\tilde{j}_2(b_\alpha). \tag{A36}$$

In the important particular case where $r_\alpha = a_\alpha$ and $r'_\alpha = a_\alpha$, using relation (A22), we reduce relation (A34) to the form

$$F_{l_1 l_2}(a_\alpha, a_\alpha, \omega) = (-1)^p \frac{2\kappa}{\pi\eta} \tilde{j}_{l_{max}}(b_\alpha) \tilde{h}_{l_{min}}(b_\alpha), \quad l_2 = l_1 + 2p \geq 0, \quad p = 0, \pm 1. \tag{A37}$$

We also need the explicit form for the quantities $C_{l_1 l_2, l}(r_\alpha, a_\beta, R_{\alpha\beta})$ and $C_{l_1 l_2, l}(r_\alpha, a_\alpha)$ for $r_\alpha = 0$. Setting $r_\alpha = 0$ in (3.40) and using (A17), we obtain

$$C_{l_1 l_2, l}(0, a_\beta, R_{\alpha\beta}) = \delta_{l_1,0}\, \delta_{l, l_2+1} \frac{\sigma_{\beta\alpha}^{l_2}}{R_{\alpha\beta}^2}, \qquad l = l_2 \pm 1 \geq 0. \tag{A38}$$

Note that this result can be also obtained from the general relation (A23) for $C_{l_1 l_2, l}(r_\alpha, r_\beta, R_{\alpha\beta})$ found for $r_\alpha > 0$ if we set $r_\beta = a_\beta$ in it and pass to the limit as $r_\alpha \to 0$.

To determine the quantity $C_{l_1 l_2}(r_\alpha, a_\alpha)$ defined by relation (3.66), where $l_2 = l_1 \pm 1 \geq 0$, for $r_\alpha = 0$, we, first, find the quantity $C_{l_1 l_2}(r_\alpha, a_\alpha)$ for $r_\alpha < a_\alpha$. To this end, using relation (A17), we get

$$C_{l_1 l_2}(r_\alpha, a_\alpha) = \delta_{l_2, l_1+1} \frac{1}{a_\alpha^2} \left(\frac{r_\alpha}{a_\alpha} \right)^{l_1}, \qquad l_2 = l_1 \pm 1 \geq 0. \tag{A39}$$

Passing in (A39) to the limit as $r_\alpha \to 0$, we obtain

$$\lim_{r_\alpha \to 0} C_{l_1 l_2}(r_\alpha, a_\alpha) = \delta_{l_1,0}\, \delta_{l_2,1} \frac{1}{a_\alpha^2}. \tag{A40}$$

At the same time,according to relation (3.66) for the quantity $C_{l_1 l_2}(r_\alpha, r'_\alpha)$, we obtain the following relation for $r_\alpha = 0$, $r'_\alpha = a_\alpha$, and $l_2 = l_1 \pm 1 \geq 0$:

$$C_{l_1 l_2}(0, a_\alpha) = \delta_{l_1,0}\, \delta_{l_2,1} \frac{1}{a_\alpha^2} \left(1 - \frac{2}{\pi} \sin x |_{x=\infty} \right) \tag{A41}$$

that has not any definite limit.

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